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On the Eigenvalue and Eigentensor Problem for a Tensor of Even Rank

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Abstract—In more detail as in [1], we study tensor modules of even orders and the eigenvalue and eigentensor problem for tensors of any even rank. The canonical representation of a tensor in the module $\mathbb{C}_{2p}(\Omega)$ is given. We present several statements and theorems about the eigentensors for tensors of even rank, and for adjoint, normal, Hermitian, and unitary tensors in a module of even order. We note that the eigenvalue and eigentensor problem for the tensor of elastic moduli was studied by Ya. Rykhlevskii in 1983–1984. Earlier, it was studied for tensors of any even rank by I. N. Vekua.

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1. LOCALLY HILBERT COMPLEX TENSOR MODULES

Consider some general issues of tensor algebra using elementary notions of algebra and functional analysis [1–9]. Let $\mathbb{C}_p(\Omega)$ be the set of complex tensors[1)] of rank $p \geq 0$ over $\mathbb{C}(\Omega)$, where Ω is a domain in the n-dimensional Riemannian space $\mathbb{R}^n$, $\mathbb{C}$ is the set of complex numbers, and $\mathbb{C}(\Omega)$ is the set of mappings $f : \Omega \to \mathbb{C}$. We assume that complex tensors can be represented in the form $\mathbb{W} = \mathbb{U} + i\mathbb{V}$, where $\mathbb{U}$ and $\mathbb{V}$ are real tensors. Hence $\mathbb{C}_0(\Omega)$ is the ring of scalars (tensors of rank zero). One can readily see that the set $\mathbb{C}_p(\Omega)$ is a module over the ring $\mathbb{C}_0(\Omega)$, i.e., a $\mathbb{C}_0(\Omega)$-module [1]. The number p is called the order of the module $\mathbb{C}_p(\Omega)$.

If $\mathbb{W} = \mathbb{U} + i\mathbb{V} \in \mathbb{C}_p(\Omega)$, then the complex conjugate tensor has the form $\overline{\mathbb{W}} = \mathbb{U} - i\mathbb{V} \in \mathbb{C}_p(\Omega)$. In what follows, we assume that the elements of the module $\mathbb{C}_p(\Omega)$ are continuous tensors in the domain Ω and the passage from one coordinate system to another is performed by a homeomorphism of the class C_k, $k \geq 1$. Therefore, if a tensor belong to the class C_m $(0 \leq m < k)$ with respect to some arbitrarily chosen coordinate system, then it belongs to the same class with respect to any other coordinate system. Obviously, the fact that a tensor belongs to the class C_m $(m < k)$ is an invariant property of this tensor. For $k = \infty$, one can consider tensors of class C_∞.

1.1. Inner r-Product (r-Convolution) of Tensors

Let us present the definition of the inner r-product of tensors.

Definition 1.1. The *inner r-product* of tensors $\mathbb{A} \in \mathbb{C}_{p+r}(\Omega)$ and $\mathbb{B} \in \mathbb{C}_{q+r}(\Omega)$ is the tensor $\mathbb{D} = \mathbb{A} \overset{r}{\otimes} \mathbb{B}$ whose components are determined by the formulas[2)]

$$D^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p} = (\mathbb{A} \overset{r}{\otimes} \mathbb{B})^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p} = A^{k_1 k_2 \ldots k_r}_{i_1 i_2 \ldots i_p} B^{j_1 j_2 \ldots j_q}_{k_1 k_2 \ldots k_r}. \tag{1.1}$$

Thus, $2r$ indices are cancelled in the inner r-product of tensors. Therefore, only tensors whose rank is not less than r may participate in the inner r-product. If there is no cancellation of indices $(r = 0)$, then such a product is called the direct product of tensors.

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1) For the definition of a tensor, e.g., see [1, 3, 10–12].

2) For the definition of the convolution operation and the trace of an endomorphism, e.g., see [3].

If $p=q=0$, then we omit the symbol $\overset{r}{\otimes}$ and simply write $\mathbb{AB}$. In this case, the inner r-product is simply called the inner product. Of course, it can be expressed by the formula $\mathbb{AB} = A_{k_1k_2\ldots k_r}B^{k_1k_2\ldots k_r}$. We note that if $\mathbb{A}$ and $\mathbb{B}$ are tensors over $\mathbb{C}(\Omega)$ and the inner product $\mathbb{AB}$ is zero for any tensor $\mathbb{B}$, then $\mathbb{A} = 0$.

1.2. Local Scalar Product of Tensors. The Local Norm of a Tensor. The Angle between Two Tensors

In $\mathbb{C}_p(\Omega)$, we introduce the notion of a local scalar product of tensors. If $\mathbb{W} = \mathbb{U} + i\mathbb{V}$ and $\mathbb{W}' = \mathbb{U}' + i\mathbb{V}' \in \mathbb{C}_p(\Omega)$, then the expression

$$(\mathbb{W}, \mathbb{W}')_x \equiv \mathbb{W}\overline{\mathbb{W}}' = \mathbb{UU}' + \mathbb{VV}' + i(\mathbb{VU}' - \mathbb{UV}')$$

is called the local scalar product of the tensors $\mathbb{W}$ and $\mathbb{W}'$ at a point x of the domain Ω. Here $\mathbb{UU}'$, $\mathbb{VV}'$, $\mathbb{VU}'$, and $\mathbb{UV}'$ stand for the inner products of the corresponding real tensors. For example, $\mathbb{UU}' = \mathbb{U}_{i_1i_2\ldots i_p}\mathbb{U}'^{i_1i_2\ldots i_p}$ where the indices run over the values 1, 2, …, n. In a similar way, by (1.1), one can introduce the notion of r-local scalar product. Let $\mathbb{W} \in \mathbb{C}_{p+r}(\Omega)$, and let $\mathbb{W}' \in \mathbb{C}_{q+r}(\Omega)$. Then the r-local scalar product of $\mathbb{W}$ and $\mathbb{W}'$ is the tensor $\mathbb{D} = (\mathbb{W}, \mathbb{W}')_x^{(r)} = \mathbb{W} \overset{r}{\otimes} \overline{\mathbb{W}}'$ with components $D^{j_1j_2\ldots j_q}_{i_1i_2\ldots i_p} = W_{i_1i_2\ldots i_pk_1k_2\ldots k_r}\overline{W}'^{k_1k_2\ldots k_rj_1j_2\ldots j_q}$. Obviously, $D = (\mathbb{W}, \mathbb{W}')_x = \mathbb{W}\overline{\mathbb{W}}' = W_{k_1k_2\ldots k_r}\overline{W}'^{k_1k_2\ldots k_r}$ for $p = q = 0$.

One can readily show that the scalar product has the following properties at each point x of the domain Ω:

$$\begin{aligned}
&(\mathbb{W}, \mathbb{W}')_x = \overline{(\mathbb{W}', \mathbb{W})_x}, \quad \forall \mathbb{W},\, \mathbb{W}' \in \mathbb{C}_p(\Omega),\\
&\lambda(\mathbb{W}, \mathbb{W}')_x = (\lambda\mathbb{W}, \mathbb{W}')_x = (\mathbb{W}, \bar{\lambda}\mathbb{W}')_x, \quad \forall \mathbb{W},\, \mathbb{W}' \in \mathbb{C}_p(\Omega), \quad \forall \lambda \in \mathbb{C},\\
&(\mathbb{W} + \mathbb{W}', \mathbb{W}'')_x = (\mathbb{W}, \mathbb{W}'')_x + (\mathbb{W}', \mathbb{W}'')_x, \quad \forall \mathbb{W},\, \mathbb{W}',\, \mathbb{W}'' \in \mathbb{C}_p(\Omega),\\
&(\mathbb{W}, \mathbb{W}' + \mathbb{W}'')_x = (\mathbb{W}, \mathbb{W}')_x + (\mathbb{W}, \mathbb{W}'')_x, \quad \forall \mathbb{W},\, \mathbb{W}',\, \mathbb{W}'' \in \mathbb{C}_p(\Omega).
\end{aligned}$$

It follows from these properties that $(\mathbb{W}, \mathbb{W}')$ is a bilinear form. In addition, if $\mathbb{W} \in \mathbb{C}_p(\Omega)$, then $(\mathbb{W}, \mathbb{W})_x = \mathbb{UU} + \mathbb{VV} \geq 0$, and the equality is attained only if $\mathbb{W}(x) = 0$, $x \in \Omega$.

Definition 1.2. An nonnegative function $\|\mathbb{W}\|_x = \sqrt{(\mathbb{W}, \mathbb{W})_x} = (\mathbb{UU} + \mathbb{VV})^{1/2} \geq 0\ \forall \mathbb{W} \in \mathbb{C}_p(\Omega)$ is called the *local norm* of the tensor $\mathbb{W}$.

One can readily prove the following inequalities [1]:

$$|(\mathbb{W}, \mathbb{W}')_x| \leq \|\mathbb{W}\|_x\|\mathbb{W}'\|_x, \quad \|\mathbb{W} + \mathbb{W}'\|_x \leq \|\mathbb{W}\|_x + \|\mathbb{W}'\|_x, \quad \forall \mathbb{W},\, \mathbb{W}' \in \mathbb{C}_p(\Omega), \tag{1.2}$$

where the second inequality is called the triangle inequality.

It follows from the above that the module $\mathbb{C}_p(\Omega)$ has the property of a Hilbert space at each point of the domain Ω. In this connection, it is called a local Hilbert space (module). Obviously, $\mathbb{C}_{2p}(\Omega)$ is also a local Hilbert module of even order, which simultaneously is a unital ring and an algebra [1]. We note that general issues concerning modules and rings are considered in [4].

Definition 1.3. Tensors $\mathbb{W}, \mathbb{W}' \in \mathbb{C}_p(\Omega)$ are said to be *orthogonal* at a point x if $(\mathbb{W}, \mathbb{W}')_x = \mathbb{W}\overline{\mathbb{W}}' = 0$.

Obviously, orthogonal tensors satisfy the Pythagorean theorem

$$\|\mathbb{W} + \mathbb{W}'\|_x^2 = \|\mathbb{W}\|_x^2 + \|\mathbb{W}'\|_x^2, \quad \forall \mathbb{W},\, \mathbb{W}' \in \mathbb{C}(\Omega), \quad x \in \Omega.$$

The cosine of the angle between real tensors $\mathbb{W}$ and $\mathbb{W}'$ is determined by the formula

$$\cos\psi = \frac{(\mathbb{W}, \mathbb{W}')_x}{\|\mathbb{W}\|_x\|\mathbb{W}'\|_x}, \quad \forall \mathbb{W},\, \mathbb{W}' \in \mathbb{C}(\Omega), \quad x \in \Omega,$$

whence, taking the first inequality in (1.2) into account, we obtain $|\cos\psi| \leq 1$.

Note that, for a set of tensors of given rank, all theorems about linear dependence and independence of a set of vectors (tensors of rank 1) remain valid. For brevity, we do not consider these facts.

2. EIGENVALUE AND EIGENTENSOR PROBLEM FOR TENSORS OF RANK $2p$

We pose the following problem [1].

Let $\mathbb{A}$ be a tensor in the algebra $\mathbb{C}_{2p}(\Omega)$. The problem is to find all tensors $\mathbb{W}$ in the module $\mathbb{C}_p(\Omega)$ that satisfy the equation

$$\mathbb{A} \overset{p}{\otimes} \mathbb{W} = \lambda \mathbb{W}, \quad \text{where } \lambda \text{ is a scalar.} \tag{2.1}$$

Equation (2.1) always has a trivial solution $\mathbb{W} = 0$. In what follows, speaking about a solution of Eq. (2.1), we mean only nontrivial solutions $\mathbb{W} \neq 0$. Thus, our goal is to study the existence conditions for nontrivial solutions of Eq. (2.1) and to find methods for their construction.

If Eq. (2.1) has a solution $\mathbb{W} \in \mathbb{C}_p(\Omega)$ for some scalar λ, then λ is called an eigenvalue of the tensor $\mathbb{A}$ and $\mathbb{W}$ is called a right eigentensor[3)] corresponding to the eigenvalue λ.

The following problem can also be considered: find all tensors $\mathbb{W}'$ in the module $\mathbb{C}_p(\Omega)$ that satisfy the equation

$$\mathbb{W}' \overset{p}{\otimes} \mathbb{A} = \mu \mathbb{W}', \quad \text{where } \mu \text{ is a scalar.} \tag{2.2}$$

If Eq. (2.2) with a given scalar μ has a nontrivial solution $\mathbb{W}' \in \mathbb{C}_p(\Omega)$, then μ is called an eigenvalue of the tensor $\mathbb{A}$ and $\mathbb{W}'$ is a left eigentensor corresponding to the number μ. In what follows, we mainly deal with right eigentensors, because all similar issues for left eigentensors can be considered similarly.

Note that if $\mathbb{W}$ is a solution of Eq. (2.1) for some scalar λ, then $\alpha\mathbb{W}$, where α is an arbitrary scalar, is also its solution. Obviously, one can always choose a scalar α so as to satisfy the condition $\|\alpha\mathbb{W}\|_x = 1\ \forall x \in \Omega$. Indeed, for this it suffices to set $\alpha = (\|\mathbb{W}\|_x)^{-1}$. Thus, the solution of Eq. (2.1) can always be normalized by the condition $\|\mathbb{W}\|_x = 1$, $x \in \Omega$. Therefore, in what follows, we mean normalized solutions of Eq. (2.1).

If, for a given eigenvalue λ, Eq. (2.1) has k linearly independent solutions $\mathbb{W}_1, \ldots, \mathbb{W}_k$, then their linear combination $\alpha_1\mathbb{W}_1 + \ldots + \alpha_k\mathbb{W}_k$, where $\alpha_1, \ldots, \alpha_k$ are arbitrary scalars, is also a solution. By the Schmidt theorem [1], these solutions can be orthonormalized, and one can assume that the conditions $(\mathbb{W}_i, \mathbb{W}_j) = W_{i,k_1,\ldots,k_p} \overline{W}_{j\cdot}^{k_1,\ldots,k_p} = \delta_{ij}$ are satisfied. Let $\mathbb{W}$ be a normalized solution of Eq. (2.1) for some eigenvalue λ. By multiplying[4)] both sides of Eq. (2.1) by the complex conjugate tensor $\overline{\mathbb{W}}$, we obtain

$$\lambda = \overline{\mathbb{W}} \odot \mathbb{A} \odot \mathbb{W} = \overline{\mathbb{W}}\mathbb{A}\mathbb{W} = \overline{W}^{i_1\ldots i_p} A_{i_1\ldots i_p j_1\ldots j_p} W^{j_1\ldots j_p}. \tag{2.3}$$

The right-hand side of this relation is invariant under coordinate transformations. Hence we conclude that each eigenvalue of the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$, if it exists, is a scalar. Taking $\mathbb{W} = \mathbb{E} \odot \mathbb{W} \equiv \mathbb{E}\mathbb{W}$ (where $\mathbb{E}$ is the unit tensor [1] in the ring $\mathbb{C}_{2p}(\Omega)$) into account, we can represent Eq. (2.1) in the form $(\lambda\mathbb{E} - \mathbb{A})\mathbb{W} = 0$ ($(\mathbb{A} - \lambda\mathbb{E})\mathbb{W} = 0$) and rewrite it in components as $(\lambda E_{i_1\ldots i_p}^{j_1\ldots j_p} - A_{i_1\ldots i_p}^{j_1\ldots j_p})W_{j_1\ldots j_p} = 0$, $i_1, \ldots, i_p;\ j_1, \ldots, j_p = \overline{1, n}$, or briefly as

$$(\lambda\delta_i^j - A_{i\cdot}^{\cdot j})W_j = 0, \quad i, j = \overline{1, n^p}. \tag{2.4}$$

Hence we conclude that system (2.4) (the tensor equation (2.1)) has a nontrivial solution if and only if the condition

$$\det(\lambda\mathbb{E} - \mathbb{A}) = 0 \tag{2.5}$$

is satisfied.

The obtained equation (2.5) is invariant under coordinate transformations, because the determinant of a tensor in the module $\mathbb{C}_{2p}(\Omega)$ is an invariant [1]. Note that $\det(\lambda\mathbb{E} - \mathbb{A}) = \det(\lambda\delta_i^j - A_{i\cdot}^{\cdot j})$ is a

[3)]In [4], the notion of an eigenelement is given and invariants of endomorphism similarity are introduced. The Hamilton–Cayley theorem etc. are proved. Under the approach considered in [4], many well-known results of matrix algebra [5] can readily be generalized to the case of spaces of endomorphisms generated by tensors of rank 4. In what follows, we consider several issues related to this generalization.

[4)]In what follows, the symbol $\overset{p}{\otimes}$ of the inner r-product is either omitted or replaced by the symbol $\odot$.

determinant of order n^p. Therefore, representing Eq. (2.5) in extended form, we obtain an algebraic equation for λ of degree n^p:

$$\lambda^{n^p} - a_1\lambda^{n^p-1} + a_2\lambda^{n^p-2} + \ldots + (-1)^{n^p-1}a_{n^p-1}\lambda + (-1)^{n^p}a_{n^p} = 0, \tag{2.6}$$

where $a_1, \ldots, a_{n^p}$ are scalars that depend on the invariants of the tensor $\mathbb{A}$. One can see that, starting from (2.2), we obtain the same relations (2.5) and (2.6) for μ, which implies that $\lambda = \mu$. The left and right eigentensors have the same eigenvalues. Let us calculate a_1 and a_{n^p}, where a_1 is the coefficient of λ^{n^p-1} in the determinant $\det(\lambda\delta_i^j - A_{i\cdot}^{\cdot j})$. The parameter λ is contained, raised only to the first power, only in the diagonal entries of this determinant. Hence each summand of the determinant that contains λ^{n^p-1} has as factors at least $n^p - 1$ diagonal entries, but then the last factor must also be a diagonal entry. Thus, the coefficient of λ^{n^p-1} is equal to the coefficient of λ^{n^p-1} in the polynomial (which is a summand of the determinant) $(\lambda - A_{1\cdot}^{\cdot 1})(\lambda - A_{2\cdot}^{\cdot 2})\ldots(\lambda - A_{n^p\cdot}^{\cdot n^p})$, i.e., is equal to $(A_{1\cdot}^{\cdot 1} + A_{2\cdot}^{\cdot 2} + \ldots + A_{n^p\cdot}^{\cdot n^p})$. Hence we have $a_1 = I_1(\mathbb{A}) = \operatorname{tr}\mathbb{A} = A_{i\cdot}^{\cdot i} = A_{i_1\ldots i_p}^{i_1\ldots i_p}$, where $I_1(\mathbb{A})$ denotes the first invariant of the tensor $\mathbb{A}$ and $\operatorname{tr}\mathbb{A}$ denotes the trace of the tensor $\mathbb{A}$.

To calculate the constant term a_{n^p}, we set $\lambda = 0$ in (2.6). Then we obtain $(-1)^{n^p}a_{n^p} = \det(-\mathbb{A}) = (-1)^{n^p}\det\mathbb{A}$ and hence $a_{n^p} = \det\mathbb{A}$. The other coefficients can also be calculated, but this is a bit more complicated. We consider them later.

Definition 2.1. The tensor $\lambda\mathbb{E} - \mathbb{A}$ is called the characteristic tensor of a tensor $\mathbb{A}$. The polynomial $P(\lambda) = \det(\lambda\mathbb{E} - \mathbb{A})$ is called the characteristic polynomial, and $P(\lambda) = \det(\lambda\mathbb{E} - \mathbb{A}) = 0$ is called the characteristic equation of the tensor $\mathbb{A}$.

Note that the characteristic polynomial and the characteristic equation of the tensor $\mathbb{A}$ are independent of the choice of the coordinate system. If $\det\mathbb{A} \neq 0$, then Eq. (2.6) has no roots equal to zero. If $\det\mathbb{A} = 0$, then Eq. (2.6) has at least one zero root. Since the tensor $\mathbb{A}$ is not identically zero, it follows that Eq. (2.6) always has nonzero roots, i.e., the tensor $\mathbb{A}$ always has nonzero eigenvalues.

We assume that $\det\mathbb{A} \neq 0$ and $\lambda_1, \ldots, \lambda_{n^p}$ are roots of Eq. (2.6), which, of course, are nonzero. Among these roots, some (or all) of them may be multiple roots. Each root of an algebraic equation is counted as a root as many times as its multiplicity is. In general, the roots of Eq. (2.6) are complex. If λ is a root of Eq. (2.6) of multiplicity $k \leq n^p$, then the homogeneous system of Eqs. (2.4) has k linearly independent solutions $\mathbb{W}_1, \ldots, \mathbb{W}_k$, and all of them are tensors in the module $\mathbb{C}_p(\Omega)$. In addition, as was already said, this system of solutions can be assumed to be orthonormal.

Proposition 2.1. *If λ and λ' are two distinct eigenvalues of the tensor $\mathbb{A}$, then any two eigentensors $\mathbb{W}$ and $\mathbb{W}'$ corresponding to them are linearly independent.*

This proposition can be proved by assuming the converse. Taking this proposition into account, we can prove the following theorem by induction.

Theorem 2.1. *The eigentensors of a tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ that correspond to pairwise distinct characteristic numbers are linearly independent.*

One can readily prove the following theorem.

Theorem 2.2. *If $\mathbb{W}_1, \mathbb{W}_2, \mathbb{W}_3, \ldots$ are eigentensors of the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ that belong to $\mathbb{C}_p(\Omega)$ and correspond to the same eigenvalue λ, then their linear combination $C_1\mathbb{W}_1 + C_2\mathbb{W}_2 + C_3\mathbb{W}_3 + \ldots$ is either zero or is also an eigentensor of the tensor $\mathbb{A}$ for the same λ.*

Proof. Indeed, it follows from $\mathbb{A}\mathbb{W}_k = \lambda\mathbb{W}_k$, $k = 1, 2, 3, \ldots$, that

$$\mathbb{A}(C_1\mathbb{W}_1 + C_2\mathbb{W}_2 + C_3\mathbb{W}_3 + \ldots) = \lambda(C_1\mathbb{W}_1 + C_2\mathbb{W}_2 + C_3\mathbb{W}_3 + \ldots).$$

Hence linearly independent eigentensors corresponding to the same eigenvalue λ form a basis of the proper submodule, each of whose tensors is an eigentensor for the same λ. In particular, each eigentensor generates a one-dimensional proper submodule. We note that a linear combination of eigentensors of the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ that correspond to distinct characteristic numbers is not an eigentensor of $\mathbb{A}$ in general.

Definition 2.2. Two tensors $\mathbb{A}$ and $\mathbb{B}$ in the module $\mathbb{C}_{2p}(\Omega)$ that satisfy the relation

$$\mathbb{B} = \mathbb{T}^{-1} \odot \mathbb{A} \odot \mathbb{T}, \tag{2.7}$$

where $\mathbb{T} \in \mathbb{C}_{2p}(\Omega)$ is a nondegenerate tensor, are said to be similar.

The similarity relation between tensors in the module $\mathbb{C}_{2p}(\Omega)$ is an equivalence relation; i.e., the following three similarity properties of tensors take place: reflexivity (any tensor $\mathbb{A}$ is always similar to itself), symmetry (if $\mathbb{A}$ is similar to $\mathbb{B}$, then $\mathbb{B}$ is also similar to $\mathbb{A}$), and transitivity (if $\mathbb{A}$ is similar to $\mathbb{B}$ and $\mathbb{B}$ is similar to $\mathbb{D}$, then $\mathbb{A}$ is similar to $\mathbb{D}$).

It follows from (2.7) that similar tensors always have equal determinants. Indeed, we can see $\det\mathbb{B} = (\det\mathbb{T}^{-1})\det\mathbb{A}\det\mathbb{T} = \det\mathbb{A}$. The determinant equation $\det\mathbb{B} = \det\mathbb{A}$ is a necessary but not sufficient condition for the similarity of tensors $\mathbb{A}$ and $\mathbb{B}$.

One can readily see that (2.7) implies $\lambda\mathbb{E} - \mathbb{B} = \mathbb{T}^{-1} \odot (\lambda\mathbb{E} - \mathbb{A}) \odot \mathbb{T}$, which, in turn, implies $\det(\lambda\mathbb{E} - \mathbb{B}) = \det(\lambda\mathbb{E} - \mathbb{A})$. Thus, similar tensors have the same characteristic polynomial (the same characteristic equation) and the same eigenvalues.

It is not difficult to write out the expressions for the coefficients of the characteristic equation in terms of the eigenvalues of the tensor $\mathbb{A}$. Indeed, we use the Viète theorem [13] to write

$$\begin{aligned}
&I_1(\mathbb{A}) \equiv a_1 = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_{n^p},\\
&I_2(\mathbb{A}) \equiv a_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \ldots + \lambda_1\lambda_{n^p} + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \ldots\\
&\qquad\qquad + \lambda_2\lambda_{n^p} + \lambda_3\lambda_4 + \lambda_3\lambda_5 + \ldots + \lambda_3\lambda_{n^p} + \ldots + \lambda_{n^p-1}\lambda_{n^p},\\
&I_{n^p-1}(\mathbb{A}) \equiv a_{n^p-1} = \lambda_1\lambda_2\ldots\lambda_{n^p-1} + \lambda_1\lambda_2\ldots\lambda_{n^p-2}\lambda_{n^p} + \ldots + \lambda_2\lambda_3\ldots\lambda_{n^p},\\
&I_{n^p}(\mathbb{A}) \equiv a_{n^p} = \lambda_1\lambda_2\ldots\lambda_{n^p-1}\lambda_{n^p},
\end{aligned}$$

where $I_k(\mathbb{A})$, $k = \overline{1, n^p}$, denotes the kth-order invariant of the tensor $\mathbb{A}$. Thus, a tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ has n^p linearly independent algebraic invariants if $\det\mathbb{A} \neq 0$.

In conclusion, note that the roots $\lambda_1, \ldots, \lambda_{n^p}$ of Eq. (2.6) and only they are eigenvalues of the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$. These roots are associated with n^p eigentensors $\mathbb{W}_1, \ldots, \mathbb{W}_{n^p}$, which are linearly independent and normalized. This system of tensors is, in general, not orthonormal, but each of its subsystems that consists of eigentensors corresponding to some multiple eigenvalue is orthonormal.

2.1. Reduction to Canonical Form (to the Principal Axes) of a Tensor of Rank $2p$

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First, we note that the following statement holds.

Proposition 2.1. *If a system of tensors $\mathbb{W}_1, \ldots, \mathbb{W}_k$ is linearly independent, then the system of tensors $\overline{\mathbb{W}}_1, \ldots, \overline{\mathbb{W}}_k$ is also linearly independent.*

This proposition and the argument at the end of the preceding section imply the following statement.

Proposition 2.3. *The system of eigentensors $\mathbb{W}_1, \ldots, \mathbb{W}_{n^p}$ of a tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ and the system of tensors $\overline{\mathbb{W}}_1, \ldots, \overline{\mathbb{W}}_{n^p}$ (separately) form bases of the module $\mathbb{C}_p(\Omega)$.*

On the basis of this proposition and the theorem about the construction of a basis in a module by using bases in modules of smaller dimensions [1], one can prove the following statement.

Proposition 2.4. *The systems of multi-tensors $\mathbb{W}_i \otimes \mathbb{W}_j$, $\mathbb{W}_i \otimes \overline{\mathbb{W}}_j$, $\overline{\mathbb{W}}_i \otimes \mathbb{W}_j$, and $\overline{\mathbb{W}}_i \otimes \overline{\mathbb{W}}_j$, $i, j = \overline{1, n^p}$, where $\mathbb{W}_1, \ldots, \mathbb{W}_{n^p}$ is a system of eigentensors of the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$, are bases (each of them, separately) of the algebra $\mathbb{C}_{2p}(\Omega)$.*

According to this proposition, each tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ can be represented as

$$\mathbb{A} = A^{i\cdot}_{\cdot j}\mathbb{W}_i \otimes \overline{\mathbb{W}}^j, \quad i, j = \overline{1, n^p}. \tag{2.8}$$

Taking into account (2.8) in the equations $\mathbb{A} \odot \mathbb{W}_k = \lambda_k\mathbb{W}_k$, we obtain $A^{i\cdot}_{\cdot j}(\mathbb{W}_i \otimes \overline{\mathbb{W}}^j) \odot \mathbb{W}_k = \lambda_k\mathbb{W}_k$. Therefore, since $(\mathbb{W}_i \otimes \overline{\mathbb{W}}^j) \odot \mathbb{W}_k = \mathbb{W}_i(\overline{\mathbb{W}}^j \odot \mathbb{W}_k) = \mathbb{W}_i(\mathbb{W}_k, \mathbb{W}^j) = \mathbb{W}_i\delta^j_k$, we obtain $A^{i\cdot}_{\cdot k}\mathbb{W}_i = \lambda_k\mathbb{W}_k$.

Because of the linear independence of the tensors $\mathbb{W}_k$ ($k = \overline{1, n^p}$), this relation implies $A^{i}_{\cdot k} = \lambda_k \delta^i_k$, $i,\ k = \overline{1, n^p}$. On the basis of the latter relation, we rewrite (2.8) as

$$\mathbb{A} = \sum_{k=1}^{n^p} \lambda_k \mathbb{W}_k \otimes \overline{\mathbb{W}}^k. \tag{2.9}$$

Obviously, (2.9) can be written in the component form

$$A^{j_1 \dots j_p}_{i_1 \dots i_p} = \sum_{k=1}^{n^p} \lambda_k W_{k, i_1 \dots i_p} \overline{W}^{k, j_1 \dots j_p}. \tag{2.10}$$

We have obtained an important relation in the form (2.9) or (2.10), which reveals the structure of each of the tensors $\mathbb{A}$ in the algebra $\mathbb{C}_{2p}(\Omega)$. In particular, each tensor in the algebra $\mathbb{C}_{2p}(\Omega)$ can be expressed in terms of invariant characteristics of this tensor, i.e., in terms of the eigenvalues λ_k and the eigentensors corresponding to them. The representation of a tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ in the form (2.9) is called the reduction of this tensor to canonical form (to the principal axes). This representation plays an important role in various fields of mathematics and mechanics.

Now we consider the case in which several eigenvalues of a tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ are zero. Obviously, all eigenvalues of a tensor cannot be zero, because then the trivial case $\mathbb{A} = 0$ takes place.

One can readily see that $\det \mathbb{A}$ is equal to the product of eigenvalues of the tensor $\mathbb{A}$, i.e.,

$$\det \mathbb{A} = a_{n^p} = \lambda_1 \dots \lambda_{n^p}. \tag{2.11}$$

Obviously, (2.11) can also be obtained from the Viète theorem. It follows from (2.11) that some eigenvalues of the tensor $\mathbb{A}$ are zero if and only if $\det \mathbb{A} = 0$. Let r be the rank of the determinant $\det \mathbb{A}$ (the rank of the matrix of the components of the tensor $\mathbb{A}$), which, of course, is a scalar characteristic of the tensor $\mathbb{A}$. Then the homogeneous system of equations

$$\begin{aligned} &A^{j_1 \dots j_p}_{i_1 \dots i_p} W_{j_1 \dots j_p} = 0 \quad (A^{\cdot j}_{i} W_j = 0), \quad i,\ j = \overline{1, n^p}, \\ &i = N(i_1, \dots, i_p), \quad j = N(j_1, \dots, j_p), \quad i_1, \dots, i_p;\ j_1, \dots, j_p = \overline{1, n}, \end{aligned}$$

where $i = N(i_1, \dots, i_p)$ is the number of an element of the set of numerical sequences $i_1, \dots, i_p$, whose number of elements is equal to n^p, has r linearly independent solutions $\mathbb{W}_1, \dots, \mathbb{W}_r$. In this case, it follows from the argument at the end of the preceding section that this system of tensors, which are tensors in the module $\mathbb{C}_p(\Omega)$, is orthonormal. Obviously, in the case under study, the multiplicity of the eigenvalue $\lambda = 0$ is equal to the rank r of the determinant $\det \mathbb{A}$. We assume that $\lambda_{n^p - r + 1} = \lambda_{n^p - r + 2} = \dots = \lambda_{n^p} = 0$, and $\lambda_i \neq 0$ if $i = \overline{1, n^p - r}$. In this case, relation (2.9) takes the form $\mathbb{A} = \sum\limits_{k=1}^{n^p - r} \lambda_k \mathbb{W}_k \otimes \overline{\mathbb{W}}^k$. We note that the eigentensors $\mathbb{W}_k$, $k = \overline{1, n^p - r}$, corresponding to the nonzero eigenvalues do not form a complete basis of the module $\mathbb{C}_p(\Omega)$. To obtain a complete basis of the module $\mathbb{C}_p(\Omega)$, they must be supplemented with r tensors $\mathbb{W}'_1, \dots, \mathbb{W}'_r$, which are nontrivial solutions of the homogeneous equation $\mathbb{A} \odot \mathbb{W} = 0$.

Thus, the eigentensors of the tensor $\mathbb{A}$ of the algebra $\mathbb{C}_{2p}(\Omega)$ form a basis of the module $\mathbb{C}_p(\Omega)$, and their pairwise direct products form a basis of the algebra $\mathbb{C}_{2p}(\Omega)$.

Definition 2.3. A tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ coinciding with its adjoint ($\mathbb{A}^* = \mathbb{A}$) is said to be Hermitian (self-adjoint).

Let $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ be a self-adjoint tensor (the definition of a self-adjoint tensor is given below), i.e., let

$$A_{i_1 \dots i_p j_1 \dots j_p} = \bar{A}_{j_1 \dots j_p i_1 \dots i_p}, \quad i_1, \dots, i_p;\ j_1, \dots, j_p = \overline{1, n}. \tag{2.12}$$

Then, for any two tensors $\mathbb{W}$ and $\mathbb{W}' \in \mathbb{C}_p(\Omega)$, we obtain the relation invariant under the coordinate transformations

$$\overline{\mathbb{W}}' \mathbb{A} \mathbb{W} = \mathbb{W} \bar{\mathbb{A}} \overline{\mathbb{W}}'. \tag{2.13}$$

This implies that condition (2.12) expresses an invariant property of the tensor of the algebra $\mathbb{C}_{2p}(\Omega)$. In the case of a real tensor, the Hermitian property means the symmetry of the form $A_{i_1 \dots i_p j_1 \dots j_p} = A_{j_1 \dots j_p i_1 \dots i_p}$.

We assume that $\mathbb{A}$ is a Hermitian tensor of the algebra $\mathbb{C}_{2p}(\Omega)$, λ is its eigenvalue, and $\mathbb{W}$ is the corresponding normalized eigentensor. Then, of course, we have

$$\mathbb{A}\mathbb{W} = \lambda\mathbb{W}, \quad \bar{\mathbb{A}}\overline{\mathbb{W}} = \bar{\lambda}\overline{\mathbb{W}}. \tag{2.14}$$

Multiplying the first relation in (2.14) by $\overline{\mathbb{W}}$ and the second by $\mathbb{W}$ and then subtracting one from the other, because of (2.13), we obtain $(\lambda - \bar{\lambda})(\mathbb{W}, \mathbb{W}) = \lambda - \bar{\lambda} = \overline{\mathbb{W}}\mathbb{A}\mathbb{W} - \mathbb{W}\bar{\mathbb{A}}\overline{\mathbb{W}} = 0$, i.e., $\lambda = \bar{\lambda}$. Thus, the eigenvalues of a Hermitian tensor are real.

Let λ and λ' be two distinct eigenvalues of the Hermitian tensor $\mathbb{A}$ of the algebra $\mathbb{C}_{2p}(\Omega)$, and let $\mathbb{W}$ and $\mathbb{W}'$ be the eigentensors corresponding to them. Then the relation $\mathbb{A}\mathbb{W} = \lambda\mathbb{W}$ and $\bar{\mathbb{A}}\overline{\mathbb{W}}' = \lambda'\overline{\mathbb{W}}'$ hold. If we multiply the first of them by $\overline{\mathbb{W}}'$ and the second of them by $\mathbb{W}$ and then subtract one from the other, then, according to (2.13), we obtain $(\lambda - \lambda')\mathbb{W}\overline{\mathbb{W}}' = \overline{\mathbb{W}}'\mathbb{A}\mathbb{W} - \mathbb{W}\bar{\mathbb{A}}\overline{\mathbb{W}}' = 0$, because we have $\lambda \neq \lambda'$ by assumption, and hence $\mathbb{W}\overline{\mathbb{W}}' = (\mathbb{W}, \mathbb{W}') = 0$; i.e., $\mathbb{W}$ and $\mathbb{W}'$ are orthogonal.

Thus, if $\mathbb{A}$ is a Hermitian tensor in the algebra $\mathbb{C}_{2p}(\Omega)$, then its eigenvalues $\lambda_1, \ldots, \lambda_{n^p}$ are real and the corresponding eigentensors form an orthonormal system. In this case, $\mathbb{W}_i = \mathbb{W}^i$ and relation (2.9) becomes

$$\mathbb{A} = \sum_{k=1}^{n^p} \lambda_k \mathbb{W}_k \otimes \overline{\mathbb{W}}_k. \tag{2.15}$$

For the components of the tensor $\mathbb{A}$, we have the expression

$$A_{i_1\ldots i_p j_1\ldots j_p} = \sum_{k=1}^{n^p} \lambda_k \mathbb{W}_{k,i_1\ldots i_p} \overline{\mathbb{W}}_{k,j_1\ldots j_p}.$$

If $\mathbb{A}$ is a real Hermitian tensor, then its eigentensors are also real. Obviously, for a real Hermitian tensor, relation (2.15) becomes $\mathbb{A} = \sum\limits_{k=1}^{n^p} \lambda_k \mathbb{W}_k \otimes \mathbb{W}_k$, and for the components, we have

$$A_{i_1\ldots i_p j_1\ldots j_p} = \sum_{k=1}^{n^p} \lambda_k W_{k,i_1\ldots i_p} W_{k,j_1\ldots j_p}.$$

One can readily see that any positive integer power of a tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ can be represented by the formula $\mathbb{A}^n = \overbrace{\mathbb{A} \odot \ldots \odot \mathbb{A}}^{n}$. If a tensor is represented in the principal axes (2.9), then we have $\mathbb{A}^n = \sum\limits_{k=1}^{n^p} \lambda_k^n \mathbb{W}_k \otimes \overline{\mathbb{W}}^k$. Using (2.9), we can define any power of a tensor with any numerical exponent in the form $\mathbb{A}^\alpha = \sum\limits_{k=1}^{n^p} \lambda_k^\alpha \mathbb{W}_k \otimes \overline{\mathbb{W}}^k$. Here we assume that $\mathbb{A}^\alpha$ is defined for the values of α for which the power λ_k^α is defined. Hence if $\det \mathbb{A} \neq 0$, then $\mathbb{A}^0 = \mathbb{E}$. Moreover, in this case, we have

$$\mathbb{A}^{-1} = \sum_{k=1}^{n^p} \lambda_k^{-1} \mathbb{W}_k \otimes \overline{\mathbb{W}}^k. \tag{2.16}$$

The invariants of the inverse tensor $\mathbb{A}^{-1}$ can be expressed in terms of the invariants of the tensor $\mathbb{A}$. Indeed, by simple transformations, using (2.16), we obtain

$$I_k(\mathbb{A}^{-1}) = I_{n^p}^{-1}(\mathbb{A}) I_{n^p-k}(\mathbb{A}), \quad k = \overline{1, n^p - 1}, \quad I_{n^p}(\mathbb{A}^{-1}) = I_{n^p}^{-1}(\mathbb{A}).$$

One can readily find the invariants of any other power of the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$, i.e., $I_k(\mathbb{A}^m)$, where $k = \overline{1, n^p}$ and m is an arbitrary positive integer.

Note that for the algebraic invariants of the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ we can take $I_k(\mathbb{A})$, $k = \overline{1, n^p}$, or $I_1(\mathbb{A}^m)$, $m = \overline{1, n^p}$. One can readily find the relation between these invariants, i.e., to express one of them in terms of the others. For example, we have the formula $I_2(\mathbb{A}) = \frac{1}{2}[I_1^2(\mathbb{A}) - I_1(\mathbb{A}^2)]$, which can readily be verified by expressing the invariants in terms of the eigenvalues of the tensor $\mathbb{A}$. To obtain the other relations expressing one of the invariants in terms of the others, it is necessary to know the Hamilton–Cayley theorem for the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$, which, for brevity, we do not consider here.

Any tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ can be represented as the sum of a symmetric and skew-symmetric (antisymmetric) tensor as follows:

$$\mathbb{A} = \mathbb{A}^S + \mathbb{A}^A, \quad \mathbb{A}^S = \tfrac{1}{2}(\mathbb{A} + \mathbb{A}^T), \quad \mathbb{A}^A = \tfrac{1}{2}(\mathbb{A} - \mathbb{A}^T).$$

In a similar way, the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ can be represented as the sum of a spherical tensor and a deviator: $\mathbb{A} = (1/n^p)I_1(\mathbb{A})\mathbb{E} + \mathrm{dev}\mathbb{A}$. Hence $I_1(\mathrm{dev}\mathbb{A}) = 0$.

3. SOME THEOREMS ABOUT ADJOINT, NORMAL, HERMITIAN, AND UNITARY TENSORS OF THE MODULE $\mathbb{C}_{2p}(\Omega)$

We present the definition of the adjoint tensor.

Definition 3.1. A tensor $\mathbb{A}^*$ is said to be the adjoint of a tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ at a point $x \in \Omega$ if the following relation holds for any tensors $\mathbb{W}$ and $\mathbb{W}'$ in $\mathbb{C}_p(\Omega)$:

$$(\mathbb{A} \odot \mathbb{W}, \mathbb{W}')_x = (\mathbb{W}, \mathbb{A}^* \odot \mathbb{W}')_x. \tag{3.1}$$

One can readily see that for any tensors $\mathbb{W} \in \mathbb{C}_p(\Omega)$ and $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ we have

$$\mathbb{A} \odot \mathbb{W} = \mathbb{W} \odot \mathbb{A}^T, \quad \mathbb{W} \odot \mathbb{A} = \mathbb{A}^T \odot \mathbb{W}. \tag{3.2}$$

Then, using the first relation in (3.2), we obtain

$$(\mathbb{A} \odot \mathbb{W}, \mathbb{W}')_x = (\mathbb{W}, \bar{\mathbb{A}}^T \odot \mathbb{W}')_x. \tag{3.3}$$

Subtracting (3.3) termwise from (3.1), we obtain $(\mathbb{W}, (\mathbb{A}^* - \bar{\mathbb{A}}^T) \odot \mathbb{W}') = 0$. Since $\mathbb{W}$ and $\mathbb{W}'$ are arbitrary, this implies $\mathbb{A}^* = \bar{\mathbb{A}}^T$. Taking the relation $\mathbb{A}^* = \bar{\mathbb{A}}^T$ to be the definition, we can readily prove (3.1). Indeed, replacing $\bar{\mathbb{A}}^T$ by $\mathbb{A}^*$ on the right-hand side in (3.3), we obtain (3.1).

Hence for each tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$, there exists an adjoint tensor $\mathbb{A}^*$, and this tensor is unique.

Using the second formula in (3.2) and formula (3.1), we can readily prove the following relation $(\mathbb{W} \odot \mathbb{A}, \mathbb{W}')_x = (\mathbb{W}, \mathbb{W}' \odot \mathbb{A}^*)_x$.

Definition 3.2. A tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ is said to be normal if it commutes with its adjoint ($\mathbb{A} \odot \mathbb{A}^* = \mathbb{A}^* \odot \mathbb{A}$).

Theorem 3.1. *Any tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ can be represented as*

$$\mathbb{A} = \mathbb{A}_1 + i\mathbb{A}_2, \tag{3.4}$$

where $\mathbb{A}_1$ and $\mathbb{A}_2$ are Hermitian tensors (Hermitian components of the tensor $\mathbb{A}$). The Hermitian components are uniquely determined by the tensor $\mathbb{A}$. The tensor $\mathbb{A}$ is normal if and only if its Hermitian components $\mathbb{A}_1$ and $\mathbb{A}_2$ commute with each other.

Proof. We assume that representation (3.4) holds. Then, passing to the adjoint tensors on both sides in (3.4), we obtain

$$\mathbb{A}^* = \mathbb{A}_1 - i\mathbb{A}_2. \tag{3.5}$$

Treating (3.4) and (3.5) as a system of equations for $\mathbb{A}_1$ and $\mathbb{A}_2$ and solving this system, we find the tensors

$$\mathbb{A}_1 = \tfrac{1}{2}(\mathbb{A} + \mathbb{A}^*), \quad \mathbb{A}_2 = \tfrac{1}{2i}(\mathbb{A} - \mathbb{A}^*), \tag{3.6}$$

which are obviously Hermitian. Conversely, determining the Hermitian tensors by using (3.6), one can readily see that they are related to $\mathbb{A}$ by (3.4).

Let $\mathbb{A}$ be a normal tensor. Then it follows from (3.6) that $\mathbb{A}_1\mathbb{A}_2 = \mathbb{A}_2\mathbb{A}_1$. From $\mathbb{A}_1\mathbb{A}_2 = \mathbb{A}_2\mathbb{A}_1$, using (3.4) and (3.5), one can obtain $\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{A}$. The proof of the theorem is complete.

Definition 3.3. We say that a tensor $\mathbb{W} \in \mathbb{C}_p(\Omega)$ is orthogonal to a subset $\mathbb{C}'_p(\Omega) \subset \mathbb{C}_p(\Omega)$ and write $\mathbb{W} \perp \mathbb{C}'_p(\Omega)$ if it is orthogonal to any tensor in $\mathbb{C}'_p(\Omega)$.

Definition 3.4. We say that two subsets $\mathbb{C}'_p(\Omega)$ and $\mathbb{C}''_p(\Omega)$ of the module $\mathbb{C}_p(\Omega)$ are mutually orthogonal and write $\mathbb{C}'_p(\Omega) \perp \mathbb{C}''_p(\Omega)$ if any tensor in one of the subsets is orthogonal to any tensor in the other subset.

Note that the orthogonality of tensors and sets of tensors can be considered both at an arbitrary point $x \in \Omega$ and in the domain Ω. The mutually orthogonal subsets $\mathbb{C}'_p(\Omega)$ and $\mathbb{C}''_p(\Omega)$ of the module $\mathbb{C}_p(\Omega)$ are submodules, and each tensor $\mathbb{W} \in \mathbb{C}_p(\Omega)$ can be uniquely represented as the sum $\mathbb{W} = \mathbb{W}' + \mathbb{W}''$, where $\mathbb{W}' \in \mathbb{C}'_p(\Omega)$ and $\mathbb{W}'' \in \mathbb{C}''_p(\Omega)$; i.e., the splitting $\mathbb{C}_p(\Omega) = \mathbb{C}'_p(\Omega) + \mathbb{C}''_p(\Omega)$, where $\mathbb{C}'_p(\Omega) \perp \mathbb{C}''_p(\Omega)$, holds. Hence the sum of dimensions of the submodules $\mathbb{C}'_p(\Omega)$ and $\mathbb{C}''_p(\Omega)$ is equal to the dimension of the module $\mathbb{C}_p(\Omega)$. We note that, in the case under study, $\mathbb{C}'_p(\Omega)$ is said to be the orthogonal complement of $\mathbb{C}''_p(\Omega)$. Obviously, $\mathbb{C}''_p(\Omega)$ is the orthogonal complement of $\mathbb{C}'_p(\Omega)$

Definition 3.5. A submodule $\mathbb{C}'_p(\Omega) \subset \mathbb{C}_p(\Omega)$ is said to be invariant with respect to a given tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ if it follows from $\mathbb{W} \in \mathbb{C}'_p(\Omega)$ that $\mathbb{A} \odot \mathbb{W} \in \mathbb{C}'_p(\Omega)$ $(\mathbb{A}\mathbb{C}'_p(\Omega) \subset \mathbb{C}'_p(\Omega))$.

Theorem 3.2. *If a certain*[5] *submodule* $\mathbb{C}'_p(\Omega) \subset \mathbb{C}_p(\Omega)$ *is invariant under* $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$*, then its orthogonal complement* $\mathbb{C}''_p(\Omega)$ *is invariant under* $\mathbb{A}^*$.

Proof. Let $\mathbb{W}' \in \mathbb{C}'_p(\Omega)$, and let $\mathbb{W}'' \in \mathbb{C}''_p(\Omega)$. Then it follows from $\mathbb{A}\mathbb{W}' \in \mathbb{C}'_p(\Omega)$ that $(\mathbb{A}\mathbb{W}', \mathbb{A}\mathbb{W}'') = 0$. From this relation, using (3.1), we obtain $(\mathbb{W}', \mathbb{A}^*\mathbb{W}'') = 0$. Since $\mathbb{W}'$ is an arbitrary tensor in the module $\mathbb{C}'_p(\Omega)$, we have $\mathbb{A}^*\mathbb{W}'' \in \mathbb{C}''_p(\Omega)$, as was required to prove.

Definition 3.6. A tensor $\mathbb{A}$ in the module $\mathbb{C}_{2p}(\Omega)$ is called a tensor of semisimple structure if it has n^p linearly independent eigentensors, where n^p is the dimension of the module $\mathbb{C}_{2p}(\Omega)$.

The following statement holds.

Proposition. *A tensor* $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ *has a simple structure if all roots of the characteristic equation are distinct.*

We note that the converse statement is not true, i.e., there exist tensors of simple structure in the module $\mathbb{C}_{2p}(\Omega)$ whose characteristic equations have multiple roots.

Theorem 3.3. *If* $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ *is a tensor of simple structure, then the conjugate tensor* $\mathbb{A}^*$ *also has a simple structure, and in this case, the complete systems of eigentensors* $\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_{n^p}$ *and* $\mathbb{W}'_1, \mathbb{W}'_2, \ldots, \mathbb{W}'_{n^p}$ *of the tensors* $\mathbb{A}$ *and* $\mathbb{A}^*$ *can be chosen so that they are biorthogonal, i.e.,* $(\mathbb{W}_k, \mathbb{W}'_i) = \delta_{ki}$, $i, k = \overline{1, n^p}$.

Proof. Let $\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_{n^p}$ be a complete system of normalized eigentensors $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$. We take these tensors to be a basis of the module $\mathbb{C}_p(\Omega)$. Let $\mathbb{C}_p^{(k)}(\Omega)$ $(k = \overline{1, n^p})$ be the submodule of the module $\mathbb{C}_p(\Omega)$ whose generating tensors are the tensors $\{\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_{n^p}\}\backslash\{\mathbb{W}_k\}$ $(k = \overline{1, n^p})$. Hence for each k, $\mathbb{C}_p^{(k)}(\Omega)$ is an $(n^p - 1)$-dimensional invariant submodule of the module $\mathbb{C}_p(\Omega)$ with respect to $\mathbb{A}$. Then, considering the one-dimensional submodule $\mathbb{C}_p^{(k)\prime}(\Omega)$ $(k = \overline{1, n^p})$ generated by the normalized tensor $\mathbb{W}'_k \perp \mathbb{C}_p^{(k)}(\Omega)$ $(k = \overline{1, n^p})$, we can readily see that it is the one-dimensional orthogonal complement of the submodule $\mathbb{C}_p^{(k)}(\Omega)$ $(k = \overline{1, n^p})$. Then it follows from Theorem 3.2 that the submodule $\mathbb{C}_p^{(k)}(\Omega)$ is invariant under $\mathbb{A}^*$; i.e., $\mathbb{A}^*\mathbb{W}'_k = \mu_k\mathbb{W}'_k$ and $\mathbb{W}' \neq 0$, $k = \overline{1, n^p}$. It follows from $\mathbb{W}'_k \perp \mathbb{C}_p^{(k)}(\Omega)$ that $(\mathbb{W}_k, \mathbb{W}'_k) = 1 \neq 0$, because otherwise the tensor $\mathbb{W}'_k$ should be zero. Thus, we have $(\mathbb{W}_i, \mathbb{W}'_j) = \delta_{ij}$, $i, j = \overline{1, n^p}$. The proof of the theorem is complete.

We note that the fact that the tensors $\mathbb{W}_1, \ldots, \mathbb{W}_{n^p}$ and $\mathbb{W}'_1, \ldots, \mathbb{W}'_{n^p}$ are biorthonormal implies the linear independence of each of these systems of tensors (taken separately).

[5] The theorems considered in the present paper are similar to the theorems about linear operators in [5]. In general, if a tensor of rank $2p$ $(p > 1)$ is identified with an endomorphism of the tensor space (of the module $\mathbb{C}_p(\Omega)$), then many problems of linear operators are automatically transferred to the case of space of endomorphisms generated by tensors of even rank. The general problems of spaces of endomorphisms can be found in [3, 4].

Theorem 3.4. *A common eigentensor* $\mathbb{W} \in \mathbb{C}_p(\Omega)$ *of the tensors* $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ *and* $\mathbb{A}^* \in \mathbb{C}_{2p}(\Omega)$ *corresponds to the complex conjugate eigenvalues of these tensors.*

Proof. Let $\mathbb{W} \in \mathbb{C}_p(\Omega)$ be a common eigentensor of the tensors $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ and $\mathbb{A}^* \in \mathbb{C}_{2p}(\Omega)$, i.e., $\mathbb{A}\mathbb{W} = \lambda\mathbb{W}$ and $\mathbb{A}^*\mathbb{W} = \mu\mathbb{W}$, $\mathbb{W} \neq 0$. Then, taking these relations into account, from (3.1) with $\mathbb{W}' = \mathbb{W}$ we obtain $\lambda(\mathbb{W}, \mathbb{W}) = \bar{\mu}(\mathbb{W}, \mathbb{W})$. This implies $\lambda = \bar{\mu}$.

Theorem 3.5. *For each tensor* $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$*, there exists an* $(n^p - 1)$*-dimensional invariant submodule* $\mathbb{C}_p^{(n^p-1)}(\Omega)$.

Proof. Let $\mathbb{W}' \in \mathbb{C}_p(\Omega)$ be an eigentensor $\mathbb{A}^* \in \mathbb{C}_{2p}(\Omega)$, and let $\mathbb{C}_p^{(1)\prime}(\Omega)$ be a one-dimensional submodule of the module $\mathbb{C}_p(\Omega)$ generated by the tensor $\mathbb{W}'$. Let $\mathbb{C}_p^{(n^p-1)}(\Omega)$ be the orthonormal complement of the one-dimensional module $\mathbb{C}_p^{(1)\prime}(\Omega)$. Since $\mathbb{C}_p^{(1)\prime}(\Omega)$ is invariant under $\mathbb{A}^*$ and $\mathbb{A} = (\mathbb{A}^*)^*$, it follows by Theorem 3.2 that the submodule $\mathbb{C}_p^{(1)\prime}(\Omega) \subset \mathbb{C}_p(\Omega)$ is invariant under $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$.

Using this theorem, one can prove the existence of a submodule $\mathbb{C}_p^{n^p-2}(\Omega) \subset \mathbb{C}_p^{n^p-1}(\Omega)$ invariant under $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$. Continuing this argument, we construct a chain of n^p successively embedded invariant submodules of the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$: $\mathbb{C}_p^1(\Omega) \subset \ldots \subset \mathbb{C}_p^{(n^p)}(\Omega) \subset \mathbb{C}_p(\Omega)$. Here the subscript in parentheses is the submodule dimension.

Theorem 3.6 (Schur 'stheorem). *For any tensor* $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$*, one can construct a basis in which the tensor has a triangular form (the matrix of the tensor components is triangular).*

Proof. Let $\mathbb{W}_1$ be a normalized tensor in $\mathbb{A} \in \mathbb{C}_p^{(1)}(\Omega)$. In $\mathbb{C}_p^{(2)}(\Omega)$, we choose a normalized tensor $\mathbb{W}_2$ such that $(\mathbb{W}_1, \mathbb{W}_2) = 0$. In $\mathbb{C}_p^{(3)}(\Omega)$ we consider a normalized tensor $\mathbb{W}_3$ such that $(\mathbb{W}_1, \mathbb{W}_3) = 0$ and $(\mathbb{W}_2, \mathbb{W}_3) = 0$. Continuing this process, we construct an orthonormal basis $\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_{n^p}$ with the property that the submodule $\mathbb{C}_p^{(k)}(\Omega)$ $(k = \overline{1, n^p})$ spanned by the first k basis tensors $\mathbb{W}_1, \ldots, \mathbb{W}_k$ is invariant under the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$. Let $\|A_{ij}\|_1^{n^p}$ be the matrix of the tensor $\mathbb{A}$. We obtain $\mathbb{A} \odot \mathbb{W}_j = \sum\limits_{i=1}^{n^p} A_{ij}\mathbb{W}_j$, where $A_{ij} = (\mathbb{A} \odot \mathbb{W}_j, \mathbb{W}_i)$. Since $\mathbb{A} \odot \mathbb{W}_j$ belongs to $\mathbb{C}_p^{(j)}(\Omega)$, we have $A_{ij} = (\mathbb{A} \odot \mathbb{W}_j, \mathbb{W}_i) = 0$ for $i > j$. Hence the matrix of the tensor components is upper triangular, and the tensor $\mathbb{A}$ has the representation $\mathbb{A} = \sum\limits_{i,j=1}^{n^p} A_{ij}\mathbb{W}_i \otimes \overline{\mathbb{W}}_j$, where $A_{ij} = 0$ for $i > j$.

We note that this theorem can readily be proved by using the general theorem on the reduction of a tensor to the Jordan form and the subsequent orthogonalization of the Jordan basis. But the above proof is, in fact, based only on the existence of an eigentensor of the tensor $\mathbb{A}$.

Now let us prove a lemma about a property of commuting tensors.

Lemma. *Commuting (permutable) tensors* $\mathbb{A}$ *and* $\mathbb{B}$ *(*$\mathbb{A} \odot \mathbb{B} = \mathbb{B} \odot \mathbb{A}$*) in the module* $\mathbb{C}_{2p}(\Omega)$ *always have a common eigentensor.*

Proof. Let $\mathbb{W} \in \mathbb{C}_p(\Omega)$ be an eigenvector of the tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$. Then $\mathbb{A} \odot \mathbb{W} = \lambda\mathbb{W}$, $\mathbb{W} \neq 0$, and since the tensors $\mathbb{A}$ and $\mathbb{B}$ in the module $\mathbb{C}_{2p}(\Omega)$ commute, we have

$$\mathbb{A} \odot \mathbb{B}^k \odot \mathbb{W} = \lambda\mathbb{B}^k \odot \mathbb{W}, \quad k = 0,\, 1,\, 2,\, \ldots \tag{3.7}$$

We assume that the first m $(0 \leq m \leq n^p)$ tensors in the system of tensors $\mathbb{W}, \mathbb{B} \odot \mathbb{W}, \mathbb{B}^2 \odot \mathbb{W}, \ldots$ are linearly independent, while the $(m+1)$th tensor $\mathbb{B}^m \odot \mathbb{W}$ is already a linear combination of the preceding tensors. One can readily see that the submodule $\mathbb{C}_p^{(m)}(\Omega) \subset \mathbb{C}_p(\Omega)$ generated by the system of tensors $\mathbb{W}, \mathbb{B}\mathbb{W}, \ldots, \mathbb{B}^{m-1}\mathbb{W}$ is invariant under $\mathbb{B}$. It follows that the submodule $\mathbb{C}_p^{(m)}(\Omega)$ contains an eigentensor $\mathbb{X}$ of the tensor $\mathbb{B}$; i.e., $\mathbb{B}\mathbb{X} = \mu\mathbb{X}$, $\mathbb{X} \neq 0$. On the other hand, using relations (3.7), we conclude that the tensors $\mathbb{W}, \mathbb{B}\mathbb{W}, \ldots, \mathbb{B}^{m-1}\mathbb{W}$ are eigentensors for the tensor $\mathbb{A}$ and correspond to the same eigenvalue λ. Hence, by Theorem 2.2, any linear combination of these tensors, and the tensor $\mathbb{X}$ in particular, is an eigentensor of the tensor $\mathbb{A}$, which corresponds to the eigenvalue λ. Thus, we have proved the existence of a common eigentensor for the commutative tensors $\mathbb{A}$ and $\mathbb{B}$ in the module $\mathbb{C}_{2p}(\Omega)$.

Using this lemma, we can prove the following theorem.

Theorem 3.7. *A normal tensor in the module $\mathbb{C}_{2p}(\Omega)$ always has a complete*[6] *orthonormal system of eigentensors.*

Proof. Let $\mathbb{A}$ be an arbitrary normal tensor in the module $\mathbb{C}_{2p}(\Omega)$. In the case under study, the tensors $\mathbb{A}$ and $\mathbb{A}^*$ commute with each other and hence, by the proof of the above lemma, have a common eigentensor $\mathbb{W}_1 \in \mathbb{C}_p(\Omega)$. Then, by Theorem 3.4, we have $\mathbb{A} \odot \mathbb{W}_1 = \lambda_1 \mathbb{W}_1$ and $\mathbb{A}^* \odot \mathbb{W}_1 = \bar{\lambda}_1 \mathbb{W}_1$, where $\mathbb{W}_1 \neq 0$. Let $\mathbb{C}_p^{(1)}(\Omega)$ denote a one-dimensional submodule generated by the tensor $\mathbb{W}_1$, and let $\mathbb{C}_p^{(1)\prime}(\Omega)$ denote the orthogonal complement of $\mathbb{C}_p^{(1)}(\Omega)$; i.e., $\mathbb{C}_p(\Omega) = \mathbb{C}_p^{(1)}(\Omega) + \mathbb{C}_p^{(1)\prime}(\Omega)$, where $\mathbb{C}_p^{(1)}(\Omega) \perp \mathbb{C}_p^{(1)\prime}(\Omega)$. Since $\mathbb{C}_p^{(1)}(\Omega)$ is invariant under $\mathbb{A}$ and $\mathbb{A}^*$, it follows from Theorem 3.2 that the submodule $\mathbb{C}_p^{(1)\prime}(\Omega)$ is also invariant under these tensors. Hence, in the invariant submodule $\mathbb{C}_p^{(1)\prime}(\Omega)$, the commuting tensors $\mathbb{A}$ and $\mathbb{A}^*$ have a common eigentensor $\mathbb{W}_2 \neq 0$; i.e., $\mathbb{A} \odot \mathbb{W}_2 = \lambda_2 \mathbb{W}_2$ and $\mathbb{A}^* \odot \mathbb{W}_2 = \bar{\lambda}_2 \mathbb{W}_2$. Obviously, $\mathbb{W}_1 \perp \mathbb{W}_2$. Let $\mathbb{C}_p^{(2)}(\Omega)$ denote the two-dimensional submodule of the module $\mathbb{C}_p(\Omega)$, generated by the tensors $\mathbb{W}_1$ and $\mathbb{W}_2$, and let $\mathbb{C}_p^{(2)\prime}(\Omega)$ denote the orthogonal complement of $\mathbb{C}_p^{(2)}(\Omega)$ ($\mathbb{C}_p(\Omega) = \mathbb{C}_p^{(2)}(\Omega) + \mathbb{C}_p^{(2)\prime}(\Omega)$), $\mathbb{C}_p^{(2)}(\Omega) \perp \mathbb{C}_p^{(2)\prime}(\Omega)$. Then we can prove in a similar way that there exists a common eigentensor $\mathbb{W}_3$ of the tensors $\mathbb{A}$ and $\mathbb{A}^*$ in $\mathbb{C}_p^{(2)\prime}(\Omega)$. Obviously, $\mathbb{W}_3 \perp \mathbb{W}_1$ and $\mathbb{W}_3 \perp \mathbb{W}_2$. Continuing this process, we obtain n^p pairwise orthogonal common eigentensors $\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_{n^p}$ for the tensors $\mathbb{A}$ and $\mathbb{A}^*$:

$$\mathbb{A} \odot \mathbb{W}_k = \lambda_k \mathbb{W}_k, \quad \mathbb{A}^* \odot \mathbb{W}_k = \bar{\lambda}_k \mathbb{W}_k, \quad \mathbb{W}_k \neq 0, \quad (\mathbb{W}_k, \mathbb{W}_i) = 0, \quad i \neq k, \quad i, k = \overline{1, n^p}. \quad (3.8)$$

Note that the tensors $\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_{n^p}$ can always be assumed to be normalized; i.e., $(\mathbb{W}_i, \mathbb{W}_j) = \delta_{ij}$, $i, j = \overline{1, n^p}$. The proof of the theorem is complete.

This theorem (in particular, relations (3.8)) implies the following statement.

Theorem 3.8. *If a tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ is normal, then each eigentensor of the tensor $\mathbb{A}$ is an eigentensor of the adjoint tensor $\mathbb{A}^*$; i.e., if $\mathbb{A}$ is normal, then $\mathbb{A}$ and $\mathbb{A}^*$ have the same eigentensors.*

The converse statement is also true.

Theorem 3.9. *If a tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ has a complete orthonormal system of eigentensors, then it is a normal tensor.*

Proof. Let $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ have a complete orthonormal system of eigentensors $\mathbb{W}_1, \mathbb{W}_2, \ldots, \mathbb{W}_{n^p}$; i.e.,

$$\mathbb{A} \odot \mathbb{W}_i = \lambda_i \mathbb{W}_i, \quad (\mathbb{W}_i, \mathbb{W}_j) = \delta_{ij}, \quad i, j = \overline{1, n^p}. \quad (3.9)$$

It is required to prove that $\mathbb{A}$ is a normal tensor. Indeed, we set $\mathbb{X}_k = \mathbb{A}^* \odot \mathbb{W}_k - \bar{\lambda}_k \mathbb{W}_k$. Then, by the properties of scalar multiplication, we have

$$(\mathbb{W}_i, \mathbb{X}_k) = (\mathbb{W}_i, \mathbb{A}^* \mathbb{W}_k) - \lambda_k (\mathbb{W}_i, \mathbb{W}_k) = (\mathbb{A}\mathbb{W}_i, \mathbb{W}_k) - \lambda_k (\mathbb{W}_i, \mathbb{W}_k) = (\lambda_i - \lambda_k)\delta_{ik} = 0, \quad i, k = \overline{1, n^p}.$$

It follows that $\mathbb{X}_k = \mathbb{A}^* \mathbb{W}_k - \bar{\lambda}_k \mathbb{W}_k$, $k = \overline{1, n^p}$; i.e.,

$$\mathbb{A}^* \odot \mathbb{W}_k = \bar{\lambda}_k \mathbb{W}_k, \quad (\mathbb{W}_i, \mathbb{W}_k) = \delta_{ik}, \quad i, k = \overline{1, n^p}. \quad (3.10)$$

From (3.9) and (3.10), we obtain

$$(\mathbb{A}^* \odot \mathbb{A}) \odot \mathbb{W}_i = \lambda_i \bar{\lambda}_i \mathbb{W}_i, \quad (\mathbb{A} \odot \mathbb{A}^*) \odot \mathbb{W}_i = \lambda_i \bar{\lambda}_i \mathbb{W}_i, \quad i = \overline{1, n^p},$$

which, in turn, implies $\mathbb{A} \odot \mathbb{A}^* = \mathbb{A}^* \odot \mathbb{A}$.

[6]The complete orthonormal system of tensors of a tensor in the module $\mathbb{C}_{2p}(\Omega)$ is understood as an orthonormal system of n^p tensors, where n^p is the number of measurements of the module $\mathbb{C}_{2p}(\Omega)$.

Thus, along with the "external" ($\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{A}$) characteristic, we obtain the following "inner" (spectral) characteristic of the normal tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$.

Theorem 3.10. *A tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ is normal if and only if it has a complete orthonormal system of eigentensors.*

This theorem can be reformulated as follows.

Theorem 3.11. *A tensor in the module $\mathbb{C}_{2p}(\Omega)$ is normal if and only if it has a simple structure.*

Note that if $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ is a tensor of simple structure, then it can be represented in the eigenbasis (canonical basis) as

$$\mathbb{A} = \sum_{i=1}^{n^p} \lambda_i \mathbb{W}_i \otimes \overline{\mathbb{W}}_i. \tag{3.11}$$

We see that

$$\mathbb{A}^* = \sum_{j=1}^{n^p} \bar{\lambda}_j \mathbb{W}_j \otimes \overline{\mathbb{W}}_j, \quad \mathbb{A} \odot \mathbb{A}^* = \mathbb{A}^* \odot \mathbb{A} = \sum_{k=1}^{n^p} \lambda_k \bar{\lambda}_k \mathbb{W}_k \otimes \overline{\mathbb{W}}_k, \tag{3.12}$$

and the tensor $\mathbb{A}\mathbb{A}^* = \mathbb{A}^*\mathbb{A}$ has nonnegative eigenvalues.

Theorem 3.12. *If $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ is a normal tensor, then $\mathbb{A}(\mathbb{A}^*)$ can be represented as a tensor polynomial in the tensor $\mathbb{A}^*(\mathbb{A})$. Moreover, two polynomials are determined by the characteristic numbers of the tensor $\mathbb{A}$.*

Proof. We use the Lagrange interpolation formula to determine two polynomials $F(\lambda)$ and $G(\lambda)$ from the conditions $F(\lambda_k) = \bar{\lambda}_k$ and $G(\bar{\lambda}_k) = \lambda_k$, $k = \overline{1, n^p}$, where $\lambda_1, \lambda_2, \ldots, \lambda_{n^p}$ are the eigenvalues of $\mathbb{A}$. Then from these formulas, (3.11), and the first relation in (3.12), we obtain

$$\mathbb{F}(\mathbb{A}) = \sum_{k=1}^{n^p} F(\lambda_k) \mathbb{W}_k \otimes \overline{\mathbb{W}}_k = \sum_{k=1}^{n^p} \bar{\lambda}_k \mathbb{W}_k \otimes \overline{\mathbb{W}}_k = \mathbb{A}^*,$$
$$\mathbb{G}(\mathbb{A}^*) = \sum_{k=1}^{n^p} G(\bar{\lambda}_k) \mathbb{W}_k \otimes \overline{\mathbb{W}}_k = \sum_{k=1}^{n^p} \lambda_k \mathbb{W}_k \otimes \overline{\mathbb{W}}_k = \mathbb{A}.$$

Thus, we obtain $\mathbb{A}^* = \mathbb{F}(\mathbb{A})$, $\mathbb{A} = \mathbb{G}(\mathbb{A}^*)$.

Let us prove the following theorem.

Theorem 3.13. *If $\mathbb{C}'_p(\Omega)$ is a submodule of the module $\mathbb{C}_p(\Omega)$ invariant under a normal tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ and $\mathbb{C}''_p(\Omega)$ is the orthogonal complement of $\mathbb{C}'_p(\Omega)$, then $\mathbb{C}''_p(\Omega)$ is also a submodule invariant under $\mathbb{A}$.*

Proof. Let $\mathbb{C}'_p(\Omega) \subset \mathbb{C}_p(\Omega)$ be a submodule invariant under $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ and $\mathbb{C}'_p(\Omega) + \mathbb{C}''_p(\Omega) = \mathbb{C}_p(\Omega)$, $\mathbb{C}'_p(\Omega) \perp \mathbb{C}''_p(\Omega)$. Then, by Theorem 3.2, the submodule $\mathbb{C}''_p(\Omega)$ is invariant under $\mathbb{A}^*$. But, by Theorem 3.12, we have $\mathbb{A} = \mathbb{G}(\mathbb{A}^*)$, where $G(\lambda)$ is a polynomial. Therefore, $\mathbb{C}''_p(\Omega)$ is invariant under $\mathbb{A}$. The proof of the theorem is complete.

Now we consider a Hermitian tensor. Since it is a specific form of a normal tensor, all the theorems and relations proved for normal tensors remain valid for Hermitian tensors. The following theorem expresses the "internal" characteristic of the Hermitian tensor along with its "external" ($\mathbb{A}^* = \mathbb{A}$) characteristic.

Theorem 3.14. *A tensor $\mathbb{H}$ of the module $\mathbb{C}_{2p}(\Omega)$ is Hermitian (self-adjoint) if and only if it has a complete orthonormal system of eigentensors with real eigenvalues.*

Proof. Let the tensor $\mathbb{H} \in \mathbb{C}_{2p}(\Omega)$ be a Hermitian tensor. We must prove that it has real characteristic numbers. Indeed, since the Hermitian tensor is a specific form of the normal tensor, by Theorem 3.10, it has a complete orthonormal system of eigentensors and can be represented as

$$\mathbb{H} = \sum_{i=1}^{n^p} \lambda_i \mathbb{W}_i \otimes \overline{\mathbb{W}}_i, \quad (\mathbb{W}_i, \mathbb{W}_j) = \mathbb{W}_i \odot \overline{\mathbb{W}}_j = \delta_{ij}, \quad i,\, j = \overline{1, n^p}. \tag{3.13}$$

Then, obviously, $\mathbb{H}^* = \sum\limits_{i=1}^{n_p} \bar{\lambda}_i \mathbb{W}_i \otimes \overline{\mathbb{W}}_i$, and relation $\mathbb{H}^* = \mathbb{H}$ implies that $\lambda_i = \bar{\lambda}_i$, $i = \overline{1, n^p}$. Conversely, we assume that a tensor $\mathbb{H} \in \mathbb{C}_{2p}(\Omega)$ has a complete orthonormal system of eigentensors with real eigenvalues. It is required to prove that it is a Hermitian tensor. Indeed, by Theorem 3.10, such a tensor is a normal tensor and can be represented as (3.13). From this, taking into account $\lambda_i = \bar{\lambda}_i$, we obtain $\mathbb{H}^* = \sum\limits_{i=1}^{n^p} \lambda_i \mathbb{W}_i \otimes \overline{\mathbb{W}}_i$. From this relation and (3.13), we conclude that $\mathbb{H}^* = \mathbb{H}$.

Now we consider a unitary tensor. First, we introduce the definition.

Definition 3.7. A tensor $\mathbb{A} \in \mathbb{C}_{2p}(\Omega)$ is said to be unitary if its inverse coincides with the adjoint ($\mathbb{A} \odot \mathbb{A}^* = \mathbb{E}$).

Hence the unitary tensor is a special form of a normal tensor and all the theorems and relations proved for normal tensors remain valid for unitary tensors.

A theorem for a unitary tensor in the module $\mathbb{C}_{2p}(\Omega)$, which is similar to Theorem 3.14 and which expresses its "internal" characteristic along with its "external" characteristic ($\mathbb{U} \odot \mathbb{U}^* = \mathbb{E}$), is stated as follows.

Theorem 3.15. *A tensor $\mathbb{U} \in \mathbb{C}_{2p}(\Omega)$ is unitary if and only if it has a complete orthonormal system of eigentensors with eigenvalues whose moduli are equal to unity.*

Proof. Let $\mathbb{U}$ be a unitary tensor. It is required to prove that it has a complete orthonormal system of eigentensors with characteristic numbers whose moduli are equal to unity. Indeed, since a unitary tensor is a special case of a normal tensor, it follows from Theorem 3.10 that it has a complete orhtonormal system of tensors and the representation

$$\mathbb{U} = \sum_{i=1}^{n^p} \lambda_i \mathbb{W}_i \otimes \overline{\mathbb{W}}_i, \quad (\mathbb{W}_i, \mathbb{W}_j) = \delta_{ij}, \quad i,\, j = \overline{1, n^p}, \tag{3.14}$$

which implies

$$\mathbb{U}^* = \sum_{j=1}^{n^p} \bar{\lambda}_j \mathbb{W}_j \otimes \mathbb{W}_j. \tag{3.15}$$

We use (3.14), (3.15), and the fact that the tensor $\mathbb{U}$ is unitary to obtain

$$\mathbb{U} \odot \mathbb{U}^* = \mathbb{U}^* \odot \mathbb{U} = \sum_{i=1}^{n^p} \lambda_i \bar{\lambda}_i \mathbb{W}_i \otimes \overline{\mathbb{W}}_i = \mathbb{E} = \sum_{j,k=1}^{n^p} \delta_{jk} \mathbb{W}_j \otimes \overline{\mathbb{W}}_k,$$

which implies

$$\lambda_i \bar{\lambda}_i = |\lambda_i|^2 = 1, \quad i = \overline{1, n^p}. \tag{3.16}$$

Conversely, let a tensor $\mathbb{U} \in \mathbb{C}_{2p}(\Omega)$ have a complete orthonormal system of eigentensors with characteristic numbers whose moduli are equal to unity. Then we have relations (3.14)–(3.16), which imply $\mathbb{U} \odot \mathbb{U}^* = \mathbb{U}^* \odot \mathbb{U} = \mathbb{E}$. The proof of the theorem is complete.

We note that this paper was designed by using the remarkable books [14–16].

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