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On the Construction of Linearly Independent Tensors

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Abstract—We consider various methods for constructing linearly independent isotropic, gyrotropic, orthotropic, and transversally isotropic tensors. We state assertions and theorem that permit one to construct these tensors. We find linearly independent above-mentioned tensors up to and including rank six. The components of the tensor may have no symmetry or have symmetries of various types.

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1. ON ISOTROPIC TENSORS IN $\mathbb{R}^3$

It is known [1–5] that $\underset{\sim}{\mathbf{E}} = \mathbf{r}_i\mathbf{r}^i = g_{ij}\mathbf{r}^i\mathbf{r}^j$ is the only isotropic tensor of rank 2 that can be used to represent any other isotropic tensor $\underset{\sim}{\mathbf{a}}$ of rank 2 in the form $\underset{\sim}{\mathbf{a}} = a\underset{\sim}{\mathbf{E}}$, where a is a scalar; i.e., an arbitrary isotropic tensor of rank 2 is a spherical tensor.

The tensors

$$\underset{\approx}{\mathbf{C}}_{(1)} = \underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}} = \mathbf{r}_i\mathbf{r}^i\mathbf{r}^j\mathbf{r}_j, \quad \underset{\approx}{\mathbf{C}}_{(2)} = \mathbf{r}_i\mathbf{r}_j\mathbf{r}^i\mathbf{r}^j, \quad \underset{\approx}{\mathbf{C}}_{(3)} = \mathbf{r}_i\underset{\sim}{\mathbf{E}}\mathbf{r}^i = \mathbf{r}_i\mathbf{r}_j\mathbf{r}^j\mathbf{r}^i \tag{1.1}$$

are three linearly independent (irreducible to each other) tensors of rank 4. The general expression for an arbitrary isotropic tensor of rank 4 is their linear combination

$$\underset{\approx}{\mathbf{C}} = \sum_{k=1}^{3} a_k \underset{\approx}{\mathbf{C}}_{(k)}.$$

If we pay attention to the structure of isotropic tensors of rank 2 and rank 4 in (1.1), then we easily see that they can be obtained from the corresponding multiplicative bases by pairwise convolution (contraction) of indices of the basis vectors and by exhausting all possible cases of such contraction. By way of example, let us also construct all linearly independent isotropic tensors of rank 6. The multiplicative basis of a tensor of rank 6 is $\mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}_l\mathbf{r}_m\mathbf{r}_n$. By contracting the indices pairwise arbitrarily, we obtain some isotropic tensor of rank 6. For example,

$$\mathbf{r}_i\mathbf{r}^i\mathbf{r}_k\mathbf{r}^k\mathbf{r}_m\mathbf{r}^m = \underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}}. \tag{1.2}$$

All other isotropic tensors of rank 6 can be obtained from (1.2) by permutations of basis vectors. Obviously, by rearranging the basis vectors in (1.2), we obtain $6! = 720$ permutations (isotropic tensors of rank 6) in the general case. Of these tensors, only fifteen are linearly independent (irreducible to each other) [3, 5]. To obtain these linearly independent tensors of rank 6, it suffices, for example, to consider the following tensors:

$$\mathbf{r}_i\mathbf{r}^i\mathbf{r}_k\mathbf{r}^k\mathbf{r}_m\mathbf{r}^m, \quad \mathbf{r}_i\mathbf{r}_k\mathbf{r}^i\mathbf{r}^k\mathbf{r}_m\mathbf{r}^m, \quad \mathbf{r}_i\mathbf{r}_k\mathbf{r}^k\mathbf{r}^i\mathbf{r}_m\mathbf{r}^m, \quad \mathbf{r}_i\mathbf{r}_k\mathbf{r}^k\mathbf{r}_m\mathbf{r}^i\mathbf{r}^m, \quad \mathbf{r}_i\mathbf{r}_k\mathbf{r}^k\mathbf{r}_m\mathbf{r}^m\mathbf{r}^i. \tag{1.3}$$

It is clear that the basis vector $\mathbf{r}^i$ occupies all possible positions in (1.3). Now by keeping the vectors $\mathbf{r}_i$ and $\mathbf{r}^i$ at their positions in (1.3) and by permuting the other vectors, we obtain two additional tensors

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from each tensor (1.3):

$$\begin{array}{lll} \mathbf{r}_i\mathbf{r}^i\mathbf{r}_k\mathbf{r}^k\mathbf{r}_m\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}^i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^k\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}^i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^m\mathbf{r}^k, \\ \mathbf{r}_i\mathbf{r}_k\mathbf{r}^i\mathbf{r}^k\mathbf{r}_m\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}^i\mathbf{r}_m\mathbf{r}^k\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}^i\mathbf{r}_m\mathbf{r}^m\mathbf{r}^k, \\ \mathbf{r}_i\mathbf{r}_k\mathbf{r}^k\mathbf{r}^i\mathbf{r}_m\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^i\mathbf{r}^k\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^i\mathbf{r}^m\mathbf{r}^k, \\ \mathbf{r}_i\mathbf{r}_k\mathbf{r}^k\mathbf{r}_m\mathbf{r}^i\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^k\mathbf{r}^i\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_m\mathbf{r}_k\mathbf{r}^k\mathbf{r}^i\mathbf{r}^m, \\ \mathbf{r}_i\mathbf{r}_k\mathbf{r}^k\mathbf{r}_m\mathbf{r}^m\mathbf{r}^i, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^k\mathbf{r}^m\mathbf{r}^i, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^m\mathbf{r}^k\mathbf{r}^i. \end{array} \tag{1.4}$$

One can readily see that, using the notation for isotropic tensors of rank 4 and the unit tensor of rank 2, we can rewrite (1.4) as

$$\begin{array}{lll} \underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}}\underset{\sim}{\mathbf{E}} = \underset{\approx}{\mathbf{C}}_{(1)}\underset{\sim}{\mathbf{E}} = \underset{\sim}{\mathbf{E}}\underset{\approx}{\mathbf{C}}_{(1)} & \underset{\sim}{\mathbf{E}}\underset{\approx}{\mathbf{C}}_{(2)}, & \underset{\sim}{\mathbf{E}}\underset{\approx}{\mathbf{C}}_{(3)}, \\ \underset{\approx}{\mathbf{C}}_{(2)}\underset{\sim}{\mathbf{E}}, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}^i\mathbf{r}_m\mathbf{r}^k\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}^i\underset{\sim}{\mathbf{E}}\mathbf{r}^k, \\ \underset{\approx}{\mathbf{C}}_{(3)}\underset{\sim}{\mathbf{E}}, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^i\mathbf{r}^k\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^i\mathbf{r}^m\mathbf{r}^k, \\ \mathbf{r}_i\underset{\sim}{\mathbf{E}}\mathbf{r}_m\mathbf{r}^i\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_k\mathbf{r}_m\mathbf{r}^k\mathbf{r}^i\mathbf{r}^m, & \mathbf{r}_i\mathbf{r}_m\underset{\sim}{\mathbf{E}}\mathbf{r}^i\mathbf{r}^m, \\ \mathbf{r}_i\underset{\approx}{\mathbf{C}}_{(1)}\mathbf{r}^i, & \mathbf{r}_i\underset{\approx}{\mathbf{C}}_{(2)}\mathbf{r}^i, & \mathbf{r}_i\underset{\approx}{\mathbf{C}}_{(3)}\mathbf{r}^i. \end{array} \tag{1.5}$$

Nine of the tensors (1.4) can be obtained by taking inner products of isotropic tensors of rank 4 (1.1); i.e.,

$$\underset{\sim}{\mathbb{C}}_{ij} = \underset{\approx}{\mathbf{C}}_{(i)} \cdot \underset{\approx}{\mathbf{C}}_{(j)}, \quad i, j = 1,\ 2,\ 3. \tag{1.6}$$

The other six tensors are contained as summands [3] in the tensor product of the discriminant tensor $\underset{\approx}{\mathbf{C}} = C_{ijk}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k$ by itself:

$$\underset{\approx}{\mathbf{C}}\underset{\approx}{\mathbf{C}} = \mathbf{r}^i\mathbf{r}^j\mathbf{r}^k[\mathbf{r}_i(\mathbf{r}_j\mathbf{r}_k - \mathbf{r}_k\mathbf{r}_j) + \mathbf{r}_j(\mathbf{r}_k\mathbf{r}_i - \mathbf{r}_i\mathbf{r}_k) + \mathbf{r}_k(\mathbf{r}_i\mathbf{r}_j - \mathbf{r}_j\mathbf{r}_i)]. \tag{1.7}$$

Hence an arbitrary tensor $\underset{\sim}{\mathbb{C}}$ of rank 6 can be represented as a linear combination of the tensors (1.6) and the tensors occurring as summands on the right-hand side in (1.7); i.e., the tensor $\underset{\sim}{\mathbb{C}}$ can be represented as

$$\mathbb{C} = \sum_{i,j=1}^{3} a_{ij}\underset{\approx}{\mathbf{C}}_{(i)} \cdot \underset{\approx}{\mathbf{C}}_{(j)} + \mathbf{r}^i\mathbf{r}^j\mathbf{r}^k[\mathbf{r}_i(b_1\mathbf{r}_j\mathbf{r}_k + b_2\mathbf{r}_k\mathbf{r}_j) + \mathbf{r}_j(b_3\mathbf{r}_k\mathbf{r}_i + b_4\mathbf{r}_i\mathbf{r}_k) + \mathbf{r}_k(b_5\mathbf{r}_i\mathbf{r}_j + b_6\mathbf{r}_j\mathbf{r}_i)]. \tag{1.8}$$

Next, omitting details concerning isotropic tensors in $\mathbb{R}^3$, note that the following theorems hold.

Theorem 1.1. *There exist no isotropic tensors of odd rank.*

Theorem 1.2. *Any isotropic tensor of given even rank in $\mathbb{R}^3$ can be constructed from the corresponding multiplicative basis by pairwise contraction of indices of the basis vectors.*

Theorem 1.3. *All linearly independent (irreducible to each other) isotropic tensors of given even rank are contained in the set of permutations obtained by all possible permutations of the basis vectors in any isotropic tensor of the same rank constructed from the corresponding multiplicative basis by pairwise contraction of the indices of the basis vectors.*

2. ON ORTHOTROPIC TENSORS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$. REPRESENTATIONS OF ORTHOTROPIC TENSORS OF RANK 2 AND RANK 4

We construct linearly independent orthotropic tensors of rank 2, 4, and 6 and state several theorems that permit one to construct orthotropic tensors. Let us introduce the definitions.

Definition 2.1. The group of coordinate transformations

$$x_{\alpha'} = a_\alpha x_\alpha \quad < \alpha = 1,\ 2,\ 3 >,$$

where (a_1, a_2, a_3) is one of the triples

$$(1,1,1), \quad (-1,1,1), \quad (1,-1,1), \quad (1,1,-1),$$
$$(1,-1,-1), \quad (-1,1,-1), \quad (-1,-1,1), \quad (-1,-1,-1),$$

is called the *orthotropy group in* $\mathbb{R}^3$.

Definition 2.2. The group of coordinate transformations

$$x_{\alpha'} = a_\alpha x_\alpha \quad < \alpha = 1,\, 2 >,$$

where (a_1, a_2) is one of the pairs

$$(1,1), \quad (-1,1), \quad (1,-1), \quad (-1,-1),$$

is called the *orthotropy group in* $\mathbb{R}^2$.

Note that the orthotropy group in $\mathbb{R}^3$, which is a subgroup of the full orthogonal group of coordinate transformations I, can be described with the use of the matrices [5]

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \tag{2.1}$$

the orthotropy group in $\mathbb{R}^2$ can be represented by the matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

A description of a group with the use of matrices is called a matrix representation of the group. If the number of matrices in a matrix representation of a group is finite, then the group is called a point group. Otherwise, the group is said to be infinite or continuous.

Note that the orthotropy group in $\mathbb{R}^3$ contains the inversion group, the groups of reflections in planes, and the symmetry groups with respect to axes in $\mathbb{R}^3$, and the orthotropy group in $\mathbb{R}^2$ consists of the inversion group and the groups of reflections in the axes in $\mathbb{R}^2$.

Now let us introduce the definition of an orthotropic tensor.

Definition 2.3. A tensor whose symmetry group is the orthotropy group in $\mathbb{R}^3$ (resp., $\mathbb{R}^2$) is called an *orthotropic tensor in* $\mathbb{R}^3$ (resp., $\mathbb{R}^2$).

Definition 2.4. The coordinate transformation group that does not change the values of tensor components is called the *symmetry group* of the tensor.

In what follows, we construct linearly independent orthotropic tensors of rank 2, rank 4, and rank 6. Prior to constructing these tensors, note that the number of linearly independent tensors of given rank whose symmetry group is a point group is given by the formula [4–6]

$$k = \frac{1}{N} \sum_{m=1}^{N} \chi^n(g_m), \tag{2.3}$$

where N is the order of the group (the number of matrices in a matrix representation of the group), the g_m are the matrices of the symmetry group, $\chi(g_m)$ is the character of the matrix representation of the symmetry group, and n is the rank of the tensors.

Using the results obtained in [1, 5–7], we conclude that the following theorems hold.

Theorem 2.1. *There exist no orthotropic tensors of odd rank.*

Theorem 2.2. *Orthotropic tensors of even rank $r = 2k$, where k is an arbitrary finite positive integer, can be obtained from the multiplicative basis of tensors of the same rank by using it to form a tensor consisting of $k = r/2$ pairs of basis vectors of the same name under the condition that summation over repeated indices is not performed.*

We note that the choice of pairs of basis vectors which are assigned the same name is absolutely arbitrary. In particular, the basis vectors bearing the same name can appear near each other or far from each other. For example, an orthotropic tensor of rank 6 can be obtained from the multiplicative basis $\mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}_l\mathbf{r}_m\mathbf{r}_n$ if we assign one name to $\mathbf{r}_i$ and $\mathbf{r}_j$, another name to $\mathbf{r}_k$ and $\mathbf{r}_l$, and a third name to $\mathbf{r}_m$ and $\mathbf{r}_n$. As a result, we obtain the following orthotropic tensors of rank 6:

$$\mathbf{r}_\alpha\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\gamma \quad <\alpha,\ \beta,\ \gamma = 1,\ 2,\ 3>, \tag{2.4}$$

where $\mathbf{r}_\alpha$ is an orthonormal basis.

In (2.4), the basis vectors[1)] of same name occur near each other. Of course, one could consider tensors obtained from (2.4) by an arbitrary permutation of the basis vectors, for example,

$$\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\gamma \quad <\alpha,\ \beta,\ \gamma = 1,\ 2,\ 3> . \tag{2.5}$$

In (2.5), the same-name basis vectors with names α and β are far from each other. Obviously, it follows from (2.4) and (2.5) that the pairs of same-name basis vectors can also be of same name. For example, this is true for the tensors

$$\mathbf{r}_\alpha\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\alpha, \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\beta, \quad <\alpha,\ \beta,\ \gamma = 1,\ 2,\ 3> .$$

From any permutation of an orthotropic tensor of rank 6 [for example, from (2.4)], one can obtain the entire set of orthotropic tensors of rank 6.

Obviously, from the set of all $6! = 720$ permutations of the tensors (2.4), which also contains equal orthotropic tensors, one can always choose linearly independent tensors, whose number is less than the total number of permutations. Hence everything said above about a tensor of rank 6 also concerns a tensor of any even rank, which can be stated as the following theorem.

Theorem 2.3. *All linearly independent orthotropic tensors of given even rank are contained in the set of permutations of basis vectors of any orthotropic tensor of the same rank composed of the corresponding multiplicative basis whose basis vectors are assigned pairwise equal names by using lowercase Greek letters over which no summation is performed.*

Once the isotropic tensors of given even rank have been constructed, the corresponding orthotropic vectors can readily be constructed from them.

Theorem 2.4. *The orthotropic tensors of given even rank can readily be constructed from isotropic tensors of the same rank by replacing same-name basis vectors with Latin indices by the corresponding same-name basis vectors with Greek indices. (For example, $\mathbf{r}_i\mathbf{r}_k\mathbf{r}^i\mathbf{r}^k$ is replaced by the tensor $\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\beta$.)*

Note that the replacement of same-name Latin indices by same-name Greek indices is equivalent to the prohibition of summation over repeated Latin indices. Therefore, Theorem 2.4 can be briefly stated as follows.

The orthotropic tensors corresponding to isotropic tensors represented by basis vectors of an orthotropic basis can be obtained from these isotropic tensors by prohibiting summation over repeated Latin indices.

The following theorem also holds.

Theorem 2.5. *A set of orthotropic tensors containing all linearly independent orthotropic tensors can be obtained from all linearly independent isotropic tensors of given even rank constructed from the basis vectors of an orthonormal basis by prohibiting summation over repeated indices.*

[1)] The notation $<\alpha,\ \beta,\ \gamma = 1,\ 2,\ 3>$ means that $\alpha,\ \beta,\ \gamma$ take the values 1, 2, 3 and summation over repeated indices is not performed.

We use Theorem 2.5 to construct all orthotropic tensors of rank 6 from the rank 6 isotropic tensors (1.4). By the assumptions of Theorem 2.5, from (1.4) we obtain the following linearly independent orthotropic tensors of rank 6:

$$\begin{array}{l} \mathbf{r}_\alpha\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\gamma; \quad \mathbf{r}_\alpha\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\beta\mathbf{r}_\gamma, \quad \beta\neq\gamma; \quad \mathbf{r}_\alpha\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\gamma\mathbf{r}_\beta, \quad \beta\neq\gamma; \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\gamma, \quad \beta\neq\alpha; \\ \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\gamma\mathbf{r}_\beta\mathbf{r}_\gamma, \quad \alpha\neq\beta,\ \beta\neq\gamma; \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\gamma\mathbf{r}_\gamma\mathbf{r}_\beta, \quad \alpha\neq\beta,\ \beta\neq\gamma; \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\gamma\mathbf{r}_\gamma, \quad \alpha\neq\beta; \\ \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma, \quad \beta\neq\gamma,\ \gamma\neq\alpha; \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\alpha\mathbf{r}_\gamma\mathbf{r}_\beta, \quad \alpha\neq\beta,\ \beta\neq\gamma,\ \gamma\neq\alpha; \\ \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\alpha\mathbf{r}_\gamma, \quad \alpha\neq\beta,\ \gamma\neq\alpha; \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\gamma, \quad \alpha\neq\beta,\ \beta\neq\gamma,\ \gamma\neq\alpha; \\ \mathbf{r}_\alpha\mathbf{r}_\gamma\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\gamma, \quad \alpha\neq\beta,\ \beta\neq\gamma,\ \gamma\neq\alpha; \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\gamma\mathbf{r}_\alpha, \quad \alpha\neq\beta,\ \alpha\neq\gamma; \\ \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\alpha, \quad \alpha\neq\beta,\ \beta\neq\gamma,\ \gamma\neq\alpha; \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\gamma\mathbf{r}_\gamma\mathbf{r}_\beta\mathbf{r}_\alpha, \quad \alpha\neq\beta,\ \beta\neq\gamma,\ \gamma\neq\alpha; \quad \alpha,\beta,\gamma=1,2,3. \end{array} \tag{2.6}$$

From the first object in (2.6), assigning all the above values to the indices, we obtain 27 tensors; from the second object in the first row, the objects in the second row, and the first object in the fourth row, we obtain 18 tensors for each of these objects; from the objects in the third row, the second object in the fifth row, and the first object in the seventh row, we obtain 12 tensors for each of these objects; and from the remaining six tensors, we obtain 6 tensors for each of these objects. Simple computations show that the total number of linearly independent orthotropic tensors of rank 6 is equal to 183. Hence the same number is obtained by the formula used in the theory of characters of matrix representations of groups [4–6].

Note that the general expression for an orthotropic tensor of rank 6 is a linear combination of the 183 orthotropic tensors of rank 6 given by (2.6). In other words, an orthotropic tensor of rank 6 has 183 linearly independent components. Of course, if a tensor is symmetric, then the number of linearly independent components is smaller. Here we do not consider tensor symmetries.

We use the above material to consider orthotropic tensors of rank 2 and 4 in more detail.

2.1. Orthotropic Tensor of Rank 2

The tensor basis of the orthotropy group is formed by the tensors [1, 2, 4, 5, 7]:

$$\underset{\sim}{\boldsymbol{\gamma}}^{(\alpha)} = \mathbf{r}_\alpha\mathbf{r}_\alpha \quad (\gamma^{(\alpha)}_{ij} = \delta_{\alpha i}\delta_{\alpha j},\ \underset{\sim}{\boldsymbol{\gamma}}^{(\alpha)} = \gamma^{(\alpha)}_{ij}\mathbf{r}_i\mathbf{r}_j) \quad < \alpha = 1,\ 2,\ 3 > . \tag{2.7}$$

By Theorem 2.5, we obtain the same tensors (2.7) from the isotropic tensor $\underset{\sim}{\mathbf{E}} = \mathbf{r}_i\mathbf{r}^i$ of rank 2. Thus, the number of linearly independent orthotropic tensors of rank 2 is equal to three. Therefore, an arbitrary orthotropic tensor $\underset{\sim}{\mathbf{a}}$ of rank 2 is a linear combination of these tensors,

$$\underset{\sim}{\mathbf{a}} = \sum_{\alpha=1}^{3} a_{\alpha\alpha}\mathbf{r}_\alpha\mathbf{r}_\alpha = a_{11}\mathbf{r}_1\mathbf{r}_1 + a_{22}\mathbf{r}_2\mathbf{r}_2 + a_{33}\mathbf{r}_3\mathbf{r}_3. \tag{2.8}$$

In components, (2.8) can be represented as

$$a_{ij} = \sum_{\alpha=1}^{3} a_{\alpha\alpha}\gamma^{(\alpha)}_{ij} = a_{11}\delta_{1i}\delta_{1j} + a_{22}\delta_{2i}\delta_{2j} + a_{33}\delta_{3i}\delta_{3j}. \tag{2.9}$$

2.2. Orthotropic Tensor of Rank 4

By Theorem 2.5, by analogy with (2.6), all linearly independent orthotropic tensors of rank 4 can be obtained from the isotropic tensors (1.1). Indeed, proceeding just as in the construction of the tensors (2.6), from (1.1) we obtain

$$\mathbf{r}_\alpha\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta; \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\beta; \quad \mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\alpha \quad < \alpha\neq\beta >; \quad \alpha,\ \beta = 1,\ 2,\ 3. \tag{2.10}$$

All in all, we have 21 orthotropic tensors of rank 4. Of course, an arbitrary orthotropic tensor of rank 4 can be represented as a linear combination of the tensors (2.10):

$$\underset{\approx}{\mathbf{C}} = C_{ijkl}\mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}_l = \sum_{\alpha,\beta=1}^{3} C_{\alpha\alpha\beta\beta}\mathbf{r}_\alpha\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta + \sum_{\alpha\neq\beta=1}^{3} C_{\alpha\beta\alpha\beta}\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\alpha\mathbf{r}_\beta + \sum_{\alpha\neq\beta=1}^{3} C_{\alpha\beta\beta\alpha}\mathbf{r}_\alpha\mathbf{r}_\beta\mathbf{r}_\beta\mathbf{r}_\alpha. \tag{2.11}$$

Using notation (2.7), we can rewrite (2.11) in components as

$$C_{ijkl} = \sum_{\alpha,\beta=1}^{3} C_{\alpha\alpha\beta\beta}\gamma_{ij}^{(\alpha)}\gamma_{kl}^{(\beta)} + \sum_{\omega\neq\varepsilon=1}^{3} C_{\omega\varepsilon\omega\varepsilon}\gamma_{ik}^{(\omega)}\gamma_{jl}^{(\varepsilon)} + \sum_{\eta\neq\vartheta=1}^{3} C_{\eta\vartheta\vartheta\eta}\gamma_{il}^{(\eta)}\gamma_{jk}^{(\vartheta)}. \tag{2.12}$$

In expanded form, (2.12) can be represented as

$$\begin{aligned} C_{ijkl} = {} & C_{1111}\gamma_{ij}^{(1)}\gamma_{kl}^{(1)} + C_{1122}\gamma_{ij}^{(1)}\gamma_{kl}^{(2)} + C_{1133}\gamma_{ij}^{(1)}\gamma_{kl}^{(3)} + C_{2211}\gamma_{ij}^{(2)}\gamma_{kl}^{(1)} + C_{2222}\gamma_{ij}^{(2)}\gamma_{kl}^{(2)} + C_{2233}\gamma_{ij}^{(2)}\gamma_{kl}^{(3)} \\ & + C_{3311}\gamma_{ij}^{(3)}\gamma_{kl}^{(1)} + C_{3322}\gamma_{ij}^{(3)}\gamma_{kl}^{(2)} + C_{3333}\gamma_{ij}^{(3)}\gamma_{kl}^{(3)} + C_{1212}\gamma_{ik}^{(1)}\gamma_{jl}^{(2)} + C_{1221}\gamma_{il}^{(1)}\gamma_{jk}^{(2)} + C_{2112}\gamma_{il}^{(2)}\gamma_{jk}^{(1)} \\ & + C_{2121}\gamma_{ik}^{(2)}\gamma_{jl}^{(1)} + C_{1313}\gamma_{ik}^{(1)}\gamma_{jl}^{(3)} + C_{1331}\gamma_{il}^{(1)}\gamma_{jk}^{(3)} + C_{3113}\gamma_{il}^{(3)}\gamma_{jk}^{(1)} + C_{3131}\gamma_{ik}^{(3)}\gamma_{jl}^{(1)} + C_{2323}\gamma_{ik}^{(2)}\gamma_{jl}^{(3)} \\ & + C_{2332}\gamma_{il}^{(2)}\gamma_{jk}^{(3)} + C_{3223}\gamma_{il}^{(3)}\gamma_{jk}^{(2)} + C_{3232}\gamma_{ik}^{(3)}\gamma_{jl}^{(2)}. \end{aligned} \tag{2.13}$$

If the components of the tensor $\underset{\approx}{\mathbf{C}}$ have the symmetry $C_{ijkl} = C_{klij}$, then, instead of (2.13), after simple computations we obtain

$$\begin{aligned} C_{ijkl} = {} & C_{1111}\gamma_{ij}^{(1)}\gamma_{kl}^{(1)} + C_{2222}\gamma_{ij}^{(2)}\gamma_{kl}^{(2)} + C_{3333}\gamma_{ij}^{(3)}\gamma_{kl}^{(3)} + C_{1122}(\gamma_{ij}^{(1)}\gamma_{kl}^{(2)} + \gamma_{ij}^{(2)}\gamma_{kl}^{(1)}) \\ & + C_{1133}(\gamma_{ij}^{(1)}\gamma_{kl}^{(3)} + \gamma_{ij}^{(3)}\gamma_{kl}^{(1)}) + C_{2233}(\gamma_{ij}^{(2)}\gamma_{kl}^{(3)} + \gamma_{ij}^{(3)}\gamma_{kl}^{(2)}) + C_{1212}\gamma_{ik}^{(1)}\gamma_{jl}^{(2)} \\ & + C_{1221}(\gamma_{il}^{(1)}\gamma_{jk}^{(2)} + \gamma_{il}^{(2)}\gamma_{jk}^{(1)}) + C_{2121}\gamma_{ik}^{(2)}\gamma_{jl}^{(1)} + C_{1313}\gamma_{ik}^{(1)}\gamma_{jl}^{(3)} + C_{1331}(\gamma_{il}^{(1)}\gamma_{jk}^{(3)} + \gamma_{il}^{(3)}\gamma_{jk}^{(1)}) \\ & + C_{3131}\gamma_{ik}^{(3)}\gamma_{jl}^{(1)} + C_{2323}\gamma_{ik}^{(2)}\gamma_{jl}^{(3)} + C_{2332}(\gamma_{il}^{(2)}\gamma_{jk}^{(3)} + \gamma_{il}^{(3)}\gamma_{jk}^{(2)}) + C_{3232}\gamma_{ik}^{(3)}\gamma_{jl}^{(2)}. \end{aligned} \tag{2.14}$$

On the right-hand side of (2.14), we have 15 independent components; i.e., an orthotropic tensor of rank 4 has 15 independent components in the Cosserat theory.

Now if we assume that, in addition to the above symmetries, the components of the tensor $\underset{\approx}{\mathbf{C}}$ also have the symmetry $C_{ijkl} = C_{jikl}$, then from (2.14) we obtain the same representation of components of an orthotropic tensor as in the classical theory of elasticity [5, 8–11] (see also [6, 12–15]).

3. ON GYROTROPIC TENSORS IN $\mathbb{R}^2$ AND TRANSVERSALLY ISOTROPIC TENSORS IN $\mathbb{R}^3$

We consider the index restriction operation and related problems [16]. We construct gyrotropic and transversally isotropic tensors of all ranks from 2 to 6. We also present several methods for constructing transversally isotropic tensors and state several theorems and propositions.

Theorem 3.1. *Consider a three-dimensional tensor $\mathbb{A}$ of arbitrary rank with components $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ with respect to the coordinate transformation group*

$$x^{i'} = x^{i'}(x^1, x^2, x^3). \tag{3.1}$$

If all lowercase Latin indices with values 1, 2, and 3 are replaced by the corresponding capital Latin indices with values 1 and 2, then we obtain an extensive tensor $\mathbb{A}$ with components $A^{I_1 I_2 \ldots I_p}_{J_1 J_2 \ldots J_q}$, which is a tensor of the same rank with respect to the coordinate transformation group

$$x^{I'} = x^{I'}(x^1, x^2, x^3), \quad x^{3'} = x^3. \tag{3.2}$$

Proof. Let us prove this theorem for tensors of rank 2. Let $A^{i}_{\cdot j}$ be the mixed components of this tensor. We have

$$A^{i'}_{\cdot j'} = D^{i'}_{i} D^{j}_{j'} A^{i}_{\cdot j}, \quad D^{i'}_{i} = \frac{\partial x^{i'}}{\partial x^{i}}, \quad D^{j}_{j'} = \frac{\partial x^{j}}{\partial x^{j'}}. \tag{3.3}$$

By setting $i' = I$ and $j' = J$ in the first relation in (3.3) and by taking into account the relations $D_3^{I'} = \partial x^{I'}/\partial x^3 = 0$ and $D_{J'}^3 = \partial x^3/\partial x^{J'} = 0$ in view of (3.2), we obtain

$$A^{I'}_{\cdot J'} = D_I^{I'} D_{J'}^J A^{I\cdot}_{\cdot J}, \quad D_I^{I'} = \frac{\partial x^{I'}}{\partial x^I}, \quad D_{J'}^J = \frac{\partial x^J}{\partial x^{J'}}. \tag{3.4}$$

Relations (3.4) complete the proof of Theorem 3.1.

Definition 3.1. The replacement of one index of a variable by another index running through a smaller (larger) set of values is called the operation of *restriction* (*extension*) of the index.

Definition 3.2. The replacement of a lowercase Latin index by a capital Latin index is called the operation of *minimal restriction* of the index.

Definition 3.3. The replacement of a lowercase Latin index by the index 3 is called the operation of *maximal restriction* of the index.

Definition 3.4. The replacement of several lowercase Latin indices of components of a three-dimensional tensor (a multiplicative basis) by capital Latin indices and the replacement of the other indices by the index 3 is called the operation of *restriction of the tensor components* (of the multiplicative basis).

Definition 3.5. The replacement of several dummy lowercase Latin indices in the representation of a three-dimensional tensor by dummy capital Latin indices and the replacement of the other indices by the index 3 is called the operation of *restriction* of the tensor.

Definition 3.6. The operation of minimal restriction of each index (umbral index) of some quantity (in the spatial tensor representation) is called the *operation of minimal restriction* of this quantity (of the spatial tensor).

Definition 3.7. The operation of maximal restriction of each index (dummy index) of some variable (in the spatial tensor representation) is called the *operation of maximal restriction* of this quantity (of the spatial tensor).

Definition 3.8. The tensor with respect to the transformation group (3.2) obtained by the operation of minimal (maximal) restriction of a spatial tensor is called the *minimal* (*maximal*) *restriction*.

Proposition 3.1. *The maximal restriction of components of a spatial tensor is a scalar with respect to the transformation group (3.2).*

Definition 3.9. The number of triples of the indices obtained by the operation of the index restriction of some variable (tensor components, a multiplicative basis) is called the *order of restriction* of this variable (tensor components, a multiplicative basis).

Proposition 3.2. *The order of restriction of a tensor is equal to the order of restriction of the tensor components or of the multiplicative basis.*

Now we can state the theorem promised above.

Theorem 3.2. *By the index restriction operation, a tensor with respect to the coordinate transformation group (3.2) is obtained from a spatial tensor.*

This more general theorem can be proved by analogy with Theorem 3.1. These theorems can readily be generalized to the case of an n-dimensional space.

One can readily see that the following statement holds.

Proposition 3.3. *A multiplicative basis (multibasis) of any order composed of the basis vectors* $\mathbf{r}_3$ *and* $\mathbf{r}^3$ *is an object (tensor) invariant under the coordinate transformation group (3.2). In addition, under the tensor multiplication of any multibasis formed of the basis vectors* $\mathbf{r}_3$ *and* $\mathbf{r}^3$ *on the left and on the right by any tensor with respect to the coordinate transformation group (3.1), objects (tensors) invariant under the coordinate transformation group (3.2) are formed. If such multibases are placed between the basis vectors in the representation of any tensor with respect to the coordinate transformation group (3.1), then a tensor with respect to the coordinate transformation group (3.2) is obtained.*

Note that the index restriction operation of order $m = r + s$, where $r \leq p$ and $s \leq q$, for the components $A^{j_1 j_2 \ldots j_q}_{i_1 i_2 \ldots i_p}$ of the tensor $\mathbb{A}$ can be performed by using the contraction of indices of these components with the indices of the product

$$g_3^{i_1} g_3^{i_2} \cdots g_3^{i_r} g_{I_{r+1}}^{i_{r+1}} g_{I_{r+2}}^{i_{r+2}} \cdots g_{I_p}^{i_p} g_{j_1}^3 g_{j_2}^3 \cdots g_{j_s}^3 g_{j_{s+1}}^{J_{s+1}} g_{j_{s+2}}^{J_{s+2}} \cdots g_{j_q}^{J_q}$$

of components of the rank 2 unit tensor.

Performing the contraction of indices $i_1, i_2, \ldots, i_r, j_1, \ldots, j_s$ of the last expression with different indices of components of a spatial tensor, we in general obtain components of different tensors with respect to the coordinate transformation group (3.2).

Note that what was said above remains true for coordinate transformation groups more special that (3.2) under the condition that the third coordinate ($x^{3'} = x^3$) remains unchanged for all coordinate transformation groups in question. In particular, instead of (3.2), one can consider the coordinate transformation group

$$x^{I'} = a^{I'\cdot}_{\cdot J} x^J, \quad x^{3'} = x^3, \tag{3.5}$$

where the $a^{I'\cdot}_{\cdot J}$ are constants, or the transversal isotropy group

$$x_{1'} = x_1 \cos\varphi + x_2 \sin\varphi, \quad x_{2'} = -x_1 \sin\varphi + x_2 \cos\varphi, \quad x_{3'} = x_3, \quad 0 \le \varphi \le 2\pi, \tag{3.6}$$

which is also called the transformation group T_3.

3.1. Two-Dimensional Gyrotropic Tensors

Let us introduce the definition of a two-dimensional gyrotropic tensor.

Definition 3.10. A two-dimensional tensor is said to be *gyrotropic* if the symmetry group of this tensor is the proper orthogonal work (rotation group) in $\mathbb{R}^2$:

$$x_{1'} = x_1 \cos\varphi + x_2 \sin\varphi, \quad x_{2'} = -x_1 \sin\varphi + x_2 \cos\varphi, \quad 0 \le \varphi \le 2\pi, \tag{3.7}$$

Note that the number of linearly independent transversally isotropic and two-dimensional gyrotropic tensors of rank n whose components do not have any symmetry is given by the formula [4–6]

$$k = \frac{1}{2\pi} \int\limits_0^{2\pi} \chi^n(g_\varphi)\, d\varphi, \tag{3.8}$$

where n is the tensor rank, $\chi(g_\varphi)$ is the character of the matrix representation of the transformation group under study, and g_φ is the matrix of the transformation group.

Note that the numbers of linearly independent transversally isotropic and two-dimensional gyrotropic tensors of rank n coincide with the numbers of the respective linearly independent components of these tensors.

In what follows, we consider various methods for constructing linearly independent two-dimensional and transversally isotropic tensors and construct these tensors up to and including rank 6.

By formula (3.8), where g_φ is the matrix of the transformation group (3.7), one can readily prove that the following theorem holds.

Theorem 3.3. *There exist no two-dimensional gyrotropic tensors of odd rank.*

Using the same formula, one can readily show that the number of linearly independent two-dimensional gyrotropic tensors of rank 0 (scalars) is equal to 1. Any other two-dimensional gyrotropic tensor is determined up to a constant factor. In a similar way, one can prove that the number of linearly independent two-dimensional gyrotropic tensors is equal to 2. Indeed, taking into account the fact that, for the transformation group (3.7), the character of the matrix representation has the form $\chi(g_\varphi) = 2\cos\varphi$, from (3.8) we readily obtain

$$k = \frac{1}{2\pi} \int\limits_0^{2\pi} (2\cos\varphi)^2 d\varphi = \frac{1}{\pi} \int\limits_0^{2\pi} (1 + \cos 2\varphi)\, d\varphi = 2.$$

The linearly independent two-dimensional gyrotropic tensors of rank 2 are the following tensors: the two-dimensional unit tensor $\underset{\sim}{\mathbf{I}} = \mathbf{r}_I\mathbf{r}^I$ of rank 2, which is simultaneously a two-dimensional isotropic tensor of rank 2, and the two-dimensional discriminant tensor (the Levi-Cività tensor) $\underset{\sim}{\mathbf{C}} = C_{IJ}\mathbf{r}^I\mathbf{r}^J$ of rank 2. Hence the general expression for the two-dimensional tensor of rank 2 whose components do not have any symmetry is a linear combination of these tensors. Now consider two-dimensional gyrotropic tensors of rank 4. In this case, the number of linearly independent tensors is equal to 6. Indeed, we have

$$k = \frac{1}{2\pi}\int_0^{2\pi}(2\cos\varphi)^4 d\varphi = \frac{1}{2\pi}\int_0^{2\pi} 2(3 + 4\cos 2\varphi + \cos 4\varphi)\,d\varphi = 6.$$

In a similar way, we can compute the number of linearly independent two-dimensional gyrotropic tensors of rank 6. It is equal to 20. Indeed, we have

$$k = \frac{1}{2\pi}\int_0^{2\pi}(2\cos\varphi)^6 d\varphi = \frac{1}{2\pi}\int_0^{2\pi} 2(10 + 15\cos 2\varphi + 6\cos 4\varphi + \cos 6\varphi)\,d\varphi = 20.$$

The number of linearly independent two-dimensional gyrotropic tensors of higher ranks can be calculated in a similar way.

Note that two-dimensional gyrotropic tensors are simultaneously transversally isotropic tensors, which will be studied below. Therefore, we do not consider their construction here.

3.2. Transversally Isotropic Tensors

Let us introduce the definition of a transversally isotropic tensor.

Definition 3.11. A tensor is said to be *transversally isotropic*, or *monotropic*, if the symmetry group of this tensor is the transversal isotropy group (3.6).

In the case under study, the number of linearly independent tensors of rank n can be found by formula (3.8), where we now have $\chi(g_\varphi) = 1 + 2\cos\varphi$, which follows from (3.6).

One can readily prove that the number of linearly independent transversally isotropic tensors is (a) equal to 1 for the set of tensors of rank 0; (b) equal to 1 for the set of tensors of rank 1; (c) equal to 3 for the set of tensors of rank 2; (d) equal to 7 for the set of tensors of rank 3; (e) equal to 19 for the set of tensors of rank 4; (f) equal to 51 for the set of tensors of rank 5; and (g) equal to 141 for the set of tensors of rank 6.

In what follows, we consider methods for constructing transversally isotropic tensors and construct linearly independent tensors up to and including rank 6.

It is well known [5] that the tensor basis of the transversal isotropy group consists of the tensors

$$\underset{\sim}{\mathbf{I}} = \mathbf{r}_I\mathbf{r}_I = \underset{\sim}{\boldsymbol{\gamma}}^1 + \underset{\sim}{\boldsymbol{\gamma}}^2, \quad \underset{\sim}{\boldsymbol{\varepsilon}} = \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_J, \quad \mathbf{r}_3 \tag{3.9}$$

or

$$\underset{\sim}{\mathbf{E}} = \mathbf{r}_i\mathbf{r}_i, \quad \underset{\sim}{\boldsymbol{\varepsilon}} = \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_j, \quad \mathbf{r}_3, \tag{3.10}$$

where $\mathbf{r}_i$ is the unit basis vector. In other words, the tensors (3.9) and (3.10) are called the generating tensors of the transversal isotropy group.

One can readily see that a transversally isotropic tensor of rank 0 is a scalar and that any other scalar is determined up to a constant factor. In a similar way, one can readily show that a transversally isotropic tensor of rank 1 is $\mathbf{r}_3$ and that the general expression for a transversally isotropic tensor of rank 1 has the form $\mathbf{a} = \lambda\mathbf{r}_3$, where λ is some number.

Since the number of linearly independent transversally isotropic tensors of rank 2 is equal to 3 and the tensors $\underset{\sim}{\mathbf{I}}$ and $\underset{\sim}{\boldsymbol{\varepsilon}}$ are two linearly independent transversally isotropic tensors of rank 2, it follows that the third tensor of rank 2 should be composed with the use of $\mathbf{r}_3$. Obviously, this is the tensor $\mathbf{r}_3\mathbf{r}_3$.

Thus, the linearly independent transversally isotropic tensors of rank 2 are the tensors

$$\underset{\sim}{\mathbf{I}} = \mathbf{r}_I\mathbf{r}_I, \quad \underset{\sim}{\boldsymbol{\varepsilon}} = \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_J, \quad \underset{\sim}{\boldsymbol{\gamma}}^3 = \mathbf{r}_3\mathbf{r}_3. \tag{3.11}$$

The general form of a transversally isotropic tensor $\mathbf{a}$ of rank 2 whose components do not have any symmetry is a linear combination of the tensors (3.11), i.e.,

$$\mathbf{a} = a\underset{\sim}{\mathbf{I}} + b\underset{\sim}{\boldsymbol{\varepsilon}} + c\mathbf{r}_3\mathbf{r}_3. \tag{3.12}$$

If $\underset{\sim}{\mathbf{a}}$ is a symmetric tensor ($\underset{\sim}{\mathbf{a}} = \underset{\sim}{\mathbf{a}}^T$), then $b = 0$, and from (3.12) we obtain the representation

$$\underset{\sim}{\mathbf{a}} = a\underset{\sim}{\mathbf{I}} + c\mathbf{r}_3\mathbf{r}_3 \quad \text{for} \quad \underset{\sim}{\mathbf{a}} = \underset{\sim}{\mathbf{a}}^T. \tag{3.13}$$

One can readily prove the following statements.

Proposition 3.4. *For any isotropic or transversally isotropic tensor $\mathbb{A}$, the tensors $\varepsilon_{IJ}\mathbf{r}_I\mathbb{A}\mathbf{r}_J$, $\mathbf{r}_I\mathbb{A}\mathbf{r}_I$, and $\mathbf{r}_3\mathbb{A}\mathbf{r}_3$ are transversally isotropic tensors.*

Proposition 3.5. *Isotropic tensors are simultaneously transversally isotropic tensors.*

Proposition 3.6. *The tensors $\mathbf{r}_I\mathbf{r}_J\mathbf{r}_I\mathbf{r}_J$, $\varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_I\mathbf{r}_L$, $\varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J\mathbf{r}_K$, and $\varepsilon_{IJ}\varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J\mathbf{r}_L$ are transversally isotropic tensors.*

One can readily see that, by using the tensors (3.9) or (3.10), by grouping them, by considering all possible cases, and by taking into account Propositions 3.4–3.6, one can construct the set of all linearly independent transversally isotropic tensors of given rank. This is one of the various methods for constructing these tensors. It will be called the grouping and searching method. Using this method for the tensors (3.9) or (3.11) and $\mathbf{r}_3$ (grouping them and taking into account Proposition 3.4), one can readily construct seven linearly independent transversally isotropic tensors:

$$\underset{\sim}{\mathbf{I}}\mathbf{r}_3, \quad \mathbf{r}_3\underset{\sim}{\mathbf{I}}, \quad \mathbf{r}_I\mathbf{r}_3\mathbf{r}_I, \quad \underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_3, \quad \mathbf{r}_3\underset{\sim}{\boldsymbol{\varepsilon}}, \quad \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J, \quad \mathbf{r}_3\mathbf{r}_3\mathbf{r}_3 = \underset{\sim}{\boldsymbol{\gamma}}\mathbf{r}_3 = \mathbf{r}_3\underset{\sim}{\boldsymbol{\gamma}}. \tag{3.14}$$

A transversally isotropic tensor of rank 3 whose components do not have any symmetry is a linear combination of the tensors (3.14).

One can readily show that three of the tensors (3.14) occur in the representation of the spatial gyrotropic tensor $\underset{\underset{\sim}{\approx}}{\boldsymbol{\varepsilon}}$ (a tensor of rank 3):

$$\underset{\underset{\sim}{\approx}}{\boldsymbol{\varepsilon}} = \underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_3 + \mathbf{r}_3\underset{\sim}{\boldsymbol{\varepsilon}} - \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J.$$

Thus, the spatial gyrotropic tensor is a special linear combination of three of seven linearly independent transversally isotropic tensors of rank 3.

We can also readily construct linearly independent transversally isotropic tensors of rank 4. These tensors can be composed by applying the grouping and searching method to the tensors (3.11) and by using Propositions 3.4–3.6:

$$\begin{gathered}
\underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}}, \quad \mathbf{r}_I\mathbf{r}_J\mathbf{r}_I\mathbf{r}_J, \quad \mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_I, \quad \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\boldsymbol{\varepsilon}}, \quad \varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_I\mathbf{r}_L, \quad \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J\mathbf{r}_K, \quad \varepsilon_{IJ}\varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J\mathbf{r}_L, \\
\varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_J, \quad \underset{\sim}{\mathbf{I}}\underset{\sim}{\boldsymbol{\varepsilon}}, \quad \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\mathbf{I}}, \quad \mathbf{r}_I\underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_I, \quad \varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_J, \quad \underset{\sim}{\mathbf{I}}\underset{\sim}{\boldsymbol{\gamma}}^3, \quad \underset{\sim}{\boldsymbol{\gamma}}^3\underset{\sim}{\mathbf{I}}, \quad \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\boldsymbol{\gamma}}^3, \quad \underset{\sim}{\boldsymbol{\gamma}}^3\underset{\sim}{\boldsymbol{\varepsilon}}, \quad \mathbf{r}_I\underset{\sim}{\boldsymbol{\gamma}}^3\mathbf{r}_J, \\
\varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\boldsymbol{\gamma}}^3\mathbf{r}_J, \quad \underset{\sim}{\boldsymbol{\gamma}}^3\underset{\sim}{\boldsymbol{\gamma}}^3, \quad \mathbf{r}_3\underset{\sim}{\mathbf{I}}\mathbf{r}_3, \quad \mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_I, \quad \mathbf{r}_I\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3, \quad \mathbf{r}_3\underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_3, \quad \varepsilon_{IJ}\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J, \quad \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3.
\end{gathered} \tag{3.15}$$

Thus, we have constructed 25 transversally isotropic tensors of rank 4. But not all of them are linearly independent. Therefore, it is necessary to choose linearly independent tensors. As is known, their number is equal to 19, i.e., of the tensors (3.15), six tensors are superfluous. One can readily show that the tensors containing $\mathbf{r}_3$ (there are 13 of them) are linearly independent. Of the remaining 12 two-dimensional tensors, we should choose six linearly independent tensors, which, obviously, together with the other 13 tensors, form a linearly independent system. For the two-dimensional tensors, we introduce the following notation:

$$\begin{gathered}
\underset{\approx}{\mathbf{C}}_{(1)} = \underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}}, \quad \underset{\approx}{\mathbf{C}}_{(2)} = \mathbf{r}_I\mathbf{r}_J\mathbf{r}_I\mathbf{r}_J, \quad \underset{\approx}{\mathbf{C}}_{(3)} = \mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_I, \quad \underset{\approx}{\mathbf{C}}_{(4)} = \underset{\sim}{\mathbf{I}}\underset{\sim}{\boldsymbol{\varepsilon}}, \\
\underset{\approx}{\mathbf{C}}_{(5)} = \varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_I\mathbf{r}_L, \quad \underset{\approx}{\mathbf{C}}_{(6)} = \varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_J, \quad \underset{\approx}{\mathbf{a}}_{(1)} = \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\boldsymbol{\varepsilon}}, \quad \underset{\approx}{\mathbf{a}}_{(2)} = \varepsilon_{IJ}\varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J\mathbf{r}_L, \\
\underset{\approx}{\mathbf{a}}_{(3)} = \varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_J, \quad \underset{\approx}{\mathbf{a}}_{(4)} = \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\mathbf{I}}, \quad \underset{\approx}{\mathbf{a}}_{(5)} = \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J\mathbf{r}_K, \quad \underset{\approx}{\mathbf{a}}_{(6)} = \mathbf{r}_I\underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_I.
\end{gathered} \tag{3.16}$$

One can readily prove the following relations between the tensors (3.16):

$$\begin{gathered}
\underset{\approx}{\mathbf{a}}_{(1)} = \underset{\approx}{\mathbf{C}}_{(2)} - \underset{\approx}{\mathbf{C}}_{(3)}, \quad \underset{\approx}{\mathbf{a}}_{(2)} = \underset{\approx}{\mathbf{C}}_{(1)} - \underset{\approx}{\mathbf{C}}_{(3)}, \quad \underset{\approx}{\mathbf{a}}_{(3)} = \underset{\approx}{\mathbf{C}}_{(1)} - \underset{\approx}{\mathbf{C}}_{(2)}, \\
\underset{\approx}{\mathbf{a}}_{(4)} = \underset{\approx}{\mathbf{C}}_{(6)} - \underset{\approx}{\mathbf{C}}_{(5)}, \quad \underset{\approx}{\mathbf{a}}_{(5)} = \underset{\approx}{\mathbf{C}}_{(6)} - \underset{\approx}{\mathbf{C}}_{(4)}, \quad \underset{\approx}{\mathbf{a}}_{(6)} = \underset{\approx}{\mathbf{C}}_{(5)} - \underset{\approx}{\mathbf{C}}_{(4)},
\end{gathered} \tag{3.17}$$

Thus, the six relations (3.17) relate the 12 tensors (3.16). Hence we can conclude that any six linearly independent tensors in the set of 12 tensors (3.16) can be taken as the desired linearly independent tensors. In particular, for example, for the linearly independent tensors we can take the first six tensors in (3.16) or the last six.

Note that these tensors are simultaneously two-dimensional gyrotropic tensors of rank 4. Since for linearly independent two-dimensional gyrotropic (transversally isotropic) tensors we can take, for example, the tensors

$$\underset{\approx}{\mathbf{C}}_{(1)}=\underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}},\quad \underset{\approx}{\mathbf{C}}_{(2)}=\mathbf{r}_I\mathbf{r}_J\mathbf{r}_I\mathbf{r}_J,\quad \underset{\approx}{\mathbf{C}}_{(3)}=\mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_I,\quad \underset{\approx}{\mathbf{C}}_{(4)}=\underset{\sim}{\mathbf{I}}\underset{\sim}{\boldsymbol{\varepsilon}},\quad \underset{\approx}{\mathbf{C}}_{(5)}=\varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_I\mathbf{r}_L,\quad \underset{\approx}{\mathbf{C}}_{(6)}=\varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_J, \tag{3.18}$$

it follows that a two-dimensional gyrotropic (transversally isotropic) tensor $\underset{\approx}{\mathbf{C}}$ of rank 4 whose components do not have any symmetry is a linear combination of these tensors:

$$\underset{\approx}{\mathbf{C}} = C_{IJKL}\mathbf{r}^I\mathbf{r}^J\mathbf{r}^K\mathbf{r}^L = \sum_{k=1}^{6} C_k\underset{\approx}{\mathbf{C}}_{(k)}, \tag{3.19}$$

where the C_k, $k = \overline{1,6}$, are some nonzero constants. We note that the first three tensors in (3.18) are two-dimensional isotropic tensors of rank 4, which, of course, are the minimal restrictions of the corresponding spatial isotropic tensors of rank 4. Therefore, we preserve the same notation for them as for the spatial tensors. If the components of the tensor $\underset{\approx}{\mathbf{C}}$ have the symmetry $C_{IJKL} = C_{KLIJ}$, then $C_4 = C_5 = C_6 = 0$ (in what follows, we prove that these constants are zero), and in this case, instead of (3.19), we have

$$\underset{\approx}{\mathbf{C}} = C_{IJKL}\mathbf{r}^I\mathbf{r}^J\mathbf{r}^K\mathbf{r}^L = \sum_{k=1}^{3} C_k\underset{\approx}{\mathbf{C}}_{(k)}. \tag{3.20}$$

Note that (3.20) is a representation of a two-dimensional isotropic (gyrotropic) tensor of rank 4 whose components have the symmetry $C_{IJKL} = C_{KLIJ}$. Clearly, this tensor has three independent components.

Now we return to the spatial case. For the tensors containing $\mathbf{r}_3$, we introduce the notation

$$\begin{gathered}
\underset{\approx}{\mathbf{C}}_{(7)} = \underset{\sim}{\mathbf{I}}\underset{\sim}{\boldsymbol{\gamma}}^{(3)},\quad \underset{\approx}{\mathbf{C}}_{(8)} = \underset{\sim}{\boldsymbol{\varepsilon}}\underset{\sim}{\boldsymbol{\gamma}}^{(3)},\quad \underset{\approx}{\mathbf{C}}_{(9)} = \underset{\sim}{\boldsymbol{\gamma}}^{(3)}\underset{\sim}{\mathbf{I}},\quad \underset{\approx}{\mathbf{C}}_{(10)} = \underset{\sim}{\boldsymbol{\gamma}}^{(3)}\underset{\sim}{\boldsymbol{\varepsilon}},\quad \underset{\approx}{\mathbf{C}}_{(11)} = \mathbf{r}_I\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3,\\
\underset{\approx}{\mathbf{C}}_{(12)} = \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3,\quad \underset{\approx}{\mathbf{C}}_{(13)} = \mathbf{r}_I\underset{\sim}{\boldsymbol{\gamma}}^{(3)}\mathbf{r}_I,\quad \underset{\approx}{\mathbf{C}}_{(14)} = \varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\boldsymbol{\gamma}}^{(3)}\mathbf{r}_J,\quad \underset{\approx}{\mathbf{C}}_{(15)} = \mathbf{r}_3\underset{\sim}{\mathbf{I}}\mathbf{r}_3,\\
\underset{\approx}{\mathbf{C}}_{(16)} = \mathbf{r}_3\boldsymbol{\varepsilon}\mathbf{r}_3,\quad \underset{\approx}{\mathbf{C}}_{(17)} = \mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_I,\quad \underset{\approx}{\mathbf{C}}_{(18)} = \varepsilon_{IJ}\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J,\quad \underset{\approx}{\mathbf{C}}_{(19)} = \underset{\sim}{\boldsymbol{\gamma}}^{(3)}\underset{\sim}{\boldsymbol{\gamma}}^{(3)}.
\end{gathered} \tag{3.21}$$

Hence the tensors (3.18) and (3.21) exhaust all linearly independent spatial transversally isotropic tensors of rank 4. Therefore, a spatial transversally isotropic tensor of rank 4 whose components do not have any symmetry has 19 independent components and is a linear combination of (3.18) and (3.21):

$$\underset{\approx}{\mathbf{C}} = C_{ijkl}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l = \sum_{k=1}^{19} C_k\underset{\approx}{\mathbf{C}}_{(k)}, \tag{3.22}$$

where the C_k, $k = \overline{1,19}$, are some nonzero constants. We rewrite (3.22) in components,

$$\begin{aligned}
C_{ijkl} &= C_1I_{ij}I_{kl} + C_2I_{ik}I_{jl} + C_3I_{il}I_{jk} + C_4I_{ij}\varepsilon_{kl} + C_5I_{ik}\varepsilon_{jl} + C_6\varepsilon_{il}I_{jk} + C_7I_{ij}\gamma^{(3)}_{kl}\\
&\quad + C_8\varepsilon_{ij}\gamma^{(3)}_{kl} + C_9\gamma^{(3)}_{ij}I_{kl} + C_{10}\gamma^{(3)}_{ij}\varepsilon_{kl} + C_{11}I_{ik}\gamma^{(3)}_{jl} + C_{12}\varepsilon_{ik}\gamma^{(3)}_{jl} + C_{13}I_{il}\gamma^{(3)}_{jk}\\
&\quad + C_{14}\varepsilon_{il}\gamma^{(3)}_{jk} + C_{15}\gamma^{(3)}_{il}I_{jk} + C_{16}\gamma^{(3)}_{il}\varepsilon_{jk} + C_{17}\gamma^{(3)}_{ik}I_{jl} + C_{18}\gamma^{(3)}_{ik}\varepsilon_{jl} + C_{19}\gamma^{(3)}_{ij}\gamma^{(3)}_{kl},\\
I_{ij} &= \delta_{Mi}\delta_{Mj},\quad \varepsilon_{ij} = \varepsilon_{MN}\delta_{Mi}\delta_{Nj} = -\varepsilon_{ji}.
\end{aligned} \tag{3.23}$$

It readily follows from (3.23) that the following components are nonzero:

$$\begin{aligned}&C_{IJKL}=C_1\delta_{IJ}\delta_{KL}+C_2\delta_{IK}\delta_{JL}+C_3\delta_{IL}\delta_{JK}+C_4\delta_{IJ}\varepsilon_{KL}+C_5\delta_{IK}\varepsilon_{JL}+C_6\varepsilon_{IL}\delta_{JK},\\&C_{IJ33}=C_7\delta_{IJ}+C_8\varepsilon_{IJ},\quad C_{33KL}=C_9\delta_{KL}+C_{10}\varepsilon_{KL},\quad C_{I3K3}=C_{11}\delta_{IK}+C_{12}\varepsilon_{IK},\\&C_{I33L}=C_{13}\delta_{IL}+C_{14}\varepsilon_{IL},\ C_{3JK3}=C_{15}\delta_{JK}+C_{16}\varepsilon_{JK},\ C_{3J3L}=C_{17}\delta_{JL}+C_{18}\varepsilon_{JL},\\&C_{3333}=C_{19}.\end{aligned}\tag{3.24}$$

Using (3.24), we can compute that a transversally isotropic tensor of rank 4 whose components do not have any symmetry has 41 nonzero components, of which 19 are independent. From the first relation in (3.24), we obtain

$$\begin{aligned}&C_{1111}=C_{2222}=C_1+C_2+C_3, && C_{1112}=-C_{2221}=C_4+C_5+C_6,\\&C_{1122}=C_{2211}=C_1, && C_{1121}=-C_{2212}=-C_4\\&C_{1212}=C_{2121}=C_2, && C_{1211}=-C_{2122}=-C_5,\\&C_{1221}=C_{2112}=C_3, && C_{2111}=-C_{1222}=-C_6.\end{aligned}\tag{3.25}$$

Therefore, we have the relations

$$\begin{aligned}&C_{1111}=C_{2222}=C_{1122}+C_{1212}+C_{1221}=C_{2211}+C_{2121}+C_{2112},\\&C_{1112}=-C_{2221}=-(C_{1121}+C_{1211}+C_{2111})=C_{2212}+C_{2122}+C_{1222}.\end{aligned}\tag{3.26}$$

From the other relations in (3.24), we obtain

$$\begin{aligned}&C_{1133}=C_{2233}=C_7,\quad C_{1233}=-C_{2133}=C_8,\quad C_{3311}=C_{3322}=C_9,\quad C_{3312}=-C_{3321}=C_{10},\\&C_{1313}=C_{2323}=C_{11},\ C_{1323}=-C_{2313}=C_{12},\ C_{1331}=C_{2332}=C_{13},\ C_{1332}=-C_{2331}=C_{14},\\&C_{3113}=C_{3223}=C_{15},\ C_{3123}=-C_{3213}=C_{16},\ C_{3131}=C_{3232}=C_{17},\ C_{3132}=-C_{3231}=C_{18},\\&C_{3333}=C_{19}.\end{aligned}\tag{3.27}$$

Using (3.25) and (3.27), we can represent the components C_{ijkl} of the tensor $\underset{\approx}{\mathbf{C}}$ in matrix form. We have

$$(C)=\begin{pmatrix}C_{1111}&C_{1122}&C_{1133}&C_{1112}&C_{1121}&0&0&0&0\\C_{1122}&C_{1111}&C_{1133}&-C_{1121}&-C_{1112}&0&0&0&0\\C_{3311}&C_{3311}&C_{3333}&C_{3312}&-C_{3312}&0&0&0&0\\C_{1211}&C_{1222}&C_{1233}&C_{1212}&C_{1221}&0&0&0&0\\-C_{1222}&-C_{1211}&-C_{1233}&C_{1221}&C_{1212}&0&0&0&0\\0&0&0&0&0&C_{1313}&C_{1331}&C_{1323}&C_{1332}\\0&0&0&0&0&C_{3113}&C_{3131}&C_{3123}&C_{3132}\\0&0&0&0&0&-C_{1323}&-C_{1332}&C_{1313}&C_{1331}\\0&0&0&0&0&-C_{3123}&-C_{3132}&C_{3113}&C_{3131}\end{pmatrix}.\tag{3.28}$$

Now assume that the components C_{ijkl} have the symmetry $C_{ijkl}=C_{klij}$. Then, by interchanging the index pairs IJ and KL in the first relation in (3.24) and by matching the right-hand sides of the resulting relation and of the first relation in (3.24), we obtain

$$C_4(\delta_{IJ}\varepsilon_{KL}-\varepsilon_{IJ}\delta_{KL})+C_6(\varepsilon_{IL}\delta_{JK}-\delta_{IL}\varepsilon_{JK})=0.$$

Since this relation holds for any I, J, K, L, we obtain $C_4 = 0$ and $C_6 = 0$. Taking this fact into account and using the second relations in the second and third rows in (3.25) and the second relation in (3.26), we can prove that $C_5 = 0$.

Then from the first relation in (3.24) we obtain

$$C_{IJKL} = C_1\delta_{IJ}\delta_{KL} + C_2\delta_{IK}\delta_{JL} + C_3\delta_{IL}\delta_{JK},$$

which is the representation of (3.20) in components. From the remaining relations, we obtain

$$C_7 = C_9, \quad C_8 = C_{10}, \quad C_{12} = 0, \quad C_{13} = C_{15}, \quad C_{14} = -C_{16}, \quad C_{18} = 0. \tag{3.29}$$

Introducing the notation

$$\begin{gathered} a_k \equiv C_k, \quad a_4 \equiv C_7 = C_9, \quad a_5 \equiv C_8 = C_{10}, \quad a_6 \equiv C_{11}, \\ a_7 \equiv C_{13} = C_{15}, \quad a_8 \equiv C_{14} = -C_{16}, \quad a_9 \equiv C_{17}, \quad a_{10} \equiv C_{19} \end{gathered}$$

and taking into account (3.29), we can rewrite (3.23) as

$$\begin{aligned} C_{ijkl} = {} & a_1 I_{ij} I_{kl} + a_2 I_{ik} I_{jl} + a_3 I_{il} I_{jk} + a_4 (I_{ij}\gamma^{(3)}_{kl} + \gamma^{(3)}_{ij} I_{kl}) + a_5(\varepsilon_{ij}\gamma^{(3)}_{kl} + \gamma^{(3)}_{ij}\varepsilon_{kl}) \\ & + a_6 I_{ik}\gamma^{(7)}_{jl} + a_7(I_{il}\gamma^{(7)}_{jk} + \gamma^{(7)}_{il} I_{jk}) + a_8(\varepsilon_{il}\gamma^{(3)}_{jk} - \gamma^{(3)}_{il}\varepsilon_{jk}) + a_9\gamma^{(3)}_{ik} I_{jl} + a_{10}\gamma^{(3)}_{ij}\gamma^{(3)}_{kl}. \end{aligned} \tag{3.30}$$

From (3.30) or (3.24), we obtain

$$\begin{gathered} C_{IJKL} = C_{KLIJ} = a_1\delta_{IJ}\delta_{KL} + a_2\delta_{IK}\delta_{JL} + a_3\delta_{IL}\delta_{JK}, \quad C_{IJ33} = C_{33IJ} = a_4\delta_{IJ} + a_5\delta\varepsilon_{IJ}, \\ C_{I3K3} = a_6\delta_{IK}, \quad C_{I33L} = C_{3LI3} = a_7\delta_{IL} + a_8\varepsilon_{IL}, \quad C_{3J3L} = a_0\delta_{JL}, \quad C_{3333} = a_{10}. \end{gathered} \tag{3.31}$$

One can readily see that the following relations remain in (3.26):

$$C_{1111} = C_{2222} = C_{1122} + C_{1212} + C_{1221} = C_{2211} + C_{2121} + C_{2112}. \tag{3.32}$$

Thus, if the components of the tensor C are symmetric with respect to the first and last pairs of indices, then the tensor has 10 independent components. In our case, the matrix (3.28) has the form

$$(C) = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{1122} & C_{1111} & C_{1133} & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{1133} & C_{1133} & C_{3333} & C_{3312} & -C_{3312} & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{3312} & C_{1212} & C_{1221} & 0 & 0 & 0 & 0 \\ 0 & 0 & -C_{3312} & C_{1221} & C_{1212} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1313} & C_{1331} & 0 & C_{1332} \\ 0 & 0 & 0 & 0 & 0 & C_{1331} & C_{3131} & -C_{1332} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -C_{1332} & C_{1313} & C_{1331} \\ 0 & 0 & 0 & 0 & 0 & C_{1332} & 0 & C_{1331} & C_{3131} \end{pmatrix}, \tag{3.33}$$

where, by virtue of (3.32),

$$C_{1111} = C_{1122} + C_{1212} + C_{1221}. \tag{3.34}$$

We see that 29 components in the matrix (3.33) are nonzero, and 10 of them are independent.

Next, if we assume that the components of the tensor $\underset{\approx}{\mathbf{C}}$ are symmetric with respect to not only the first and last pairs of indices but also, for example, the last two indices, i.e., if $C_{ijkl} = C_{klij} = C_{ijlk}$, then from (3.31) we obtain

$$a_2 = a_3, \quad a_5 = 0, \quad a_6 = a_7 = a_9, \quad a_8 = 0. \tag{3.35}$$

Using (3.35), we conclude that in our case the tensor $\underset{\approx}{\mathbf{C}}$ has five independent components. In this case, the symmetry with respect to the first two indices of the components follows from the already existing symmetries. By introducing the notation $\Lambda_1 = a_1$, $\Lambda_2 = a_2$, $\Lambda_3 = a_4$, $\Lambda_4 = a_{10}$, and $\Lambda_5 = a_6$, we can rewrite (3.30) as

$$C_{ijkl} = C_{klij} = C_{ijlk} = \Lambda_1 I_{ij} I_{kl} + \Lambda_2 (I_{ik} I_{jl} + I_{il} I_{jk}) + \Lambda_3 (I_{ij} \gamma_{kl}^{(3)} + \gamma_{ij}^{(3)} I_{kl}) + \Lambda_4 \gamma_{ij}^{(3)} \gamma_{kl}^{(3)} + \Lambda_5 (I_{ik} \gamma_{jl}^{(3)} + I_{il} \gamma_{jk}^{(3)} + \gamma_{il}^{(3)} I_{jk} + \gamma_{ik}^{(3)} I_{jl}). \quad (3.36)$$

From (3.36) and (3.31), we obtain

$$C_{IJKL} = \Lambda_1 \delta_{IJ} \delta_{KL} + \Lambda_2 (\delta_{IK} \delta_{JL} + \delta_{IL} \delta_{JK}), \quad C_{IJ33} = \Lambda_3 \delta_{IJ}, \quad C_{3333} = \Lambda_4, \quad C_{I3K3} = \Lambda_5 \delta_{IK}.$$

In addition, from (3.32) we have $C_{1212} = \frac{1}{2}(C_{1111} - C_{1122})$. Note that (3.36) coincides with the representation of these components given in [5, 8–11] (see also [6, 12, 13, 15]).

Similarly, by using the grouping and searching method, one can construct linearly independent transversally isotropic tensors of any rank, in particular, tensors of rank 5 and 6. But in what follows we construct linearly independent tensors of rank 5 and of rank 6 by a somewhat different method, which will be called the contraction and searching method.

It follows from the structures of the tensors (3.9), (3.11), (3.14), (3.18), and (3.22) that they can also be constructed from the corresponding multiplicative bases (multibases) by using the index restriction to compose those multibases which are then used to construct the desired tensors by contracting the indices of these multibases with the indices of the two-dimensional Kronecker and Levi-Cività symbols. Note that transversally isotropic tensors of even (odd) rank can be constructed from the multibases obtained from the corresponding multibases by even (odd) orders of restriction.

What was said above can be stated as the following theorem.

Theorem 3.4. *To construct transversally isotropic tensors of even (odd) rank, it suffices to compose all possible multibases from the corresponding multibasis by using even (odd) orders of restriction. The indices of the multibases obtained by the restriction must be contracted with the indices of the two-dimensional Kronecker and Levi-Cività symbols, exhausting all possible cases. Then the set of tensors of given rank constructed by the above method contains all linearly independent transversally isotropic tensors.*

In what follows, we use this theorem to construct linearly independent transversally isotropic tensors of rank 5 and 6. First, we construct tensors of rank 5. To this end, we use the odd orders (first, third, fifth) of restriction of the multibasis $\mathbf{r}_i \mathbf{r}_j \mathbf{r}_k \mathbf{r}_l \mathbf{r}_m$ to construct all possible bases. We have

$$\begin{gathered} \mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L\mathbf{r}_3, \quad \mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_3\mathbf{r}_L, \quad \mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_K\mathbf{r}_L, \quad \mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L, \quad \mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L, \\ \mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3, \quad \mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3, \quad \mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3, \quad \mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J, \quad \mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J, \quad \mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J, \\ \mathbf{r}_2\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3, \quad \mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_J, \quad \mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3, \quad \mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_3, \quad \mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3. \end{gathered} \quad (3.37)$$

All in all, we obtain 16 multibases and, contracting them with the two-dimensional Kronecker symbol δ_{MN} and the Levi-Cività symbol ε_{ST} and exhausting all possible cases, use them to construct the set of transversally isotropic tensors of rank 5, among which there are linearly independent tensors. One can see that the number of multibases in (3.37) whose order of restriction is equal to 1 is $C_5^1 = 5$; the number of multibases whose order of restriction is equal to 3 is $C_5^3 = 10$; and (3.37) contains one multibasis $C_5^5 = 1$ (the maximal restriction of the spatial multibasis), which consists only of $\mathbf{r}_3$.

Now, to each multibasis containing only one $\mathbf{r}_3$ [the first five bases in (3.37)], we apply contractions similar to those used to compose the tensors (3.18). Then we obtain six linearly independent tensors for each of the multibases. In this case, $\mathbf{r}_3$ occupies a certain place in the representation of each of the tensors thus obtained. For example, from the first multibasis, using the above method, we obtain the tensors that can be obtained by attaching $\mathbf{r}_3$ on the right to the tensors (3.18). In a similar way, from the fifth multibasis we obtain the tensor that can be obtained by attaching $\mathbf{r}_3$ on the left to the tensors (3.18). From the second multibasis, we obtain the tensors that can be obtained by placing $\mathbf{r}_3$ at the forth place (from left to right) between the basis vectors of each of the tensors (3.18). In a similar way, from the third and fourth multibases we can construct the tensors that can be obtained by placing $\mathbf{r}_3$ at the third and second places, respectively, between the basis vectors of each of the tensors (3.18). Thus, from the first five multibases (3.37) we obtain 30 linearly independent transversally isotropic tensors of rank 5.

It is easily seen that one can construct two transversally isotropic tensors from each of the multibases containing three $\mathbf{r}_3$, by contracting each of them first with the two-dimensional Kronecker symbol and then with the two-dimensional Levi-Cività symbol. Since the number of multibases containing three $\mathbf{r}_3$ is equal to 10, we obtain all in all 20 linearly independent tensors of rank 5. The last multibasis in (3.37) (the maximal restriction) is a transversally isotropic tensor of rank 5.

Thus, from (3.37) by using the contraction and searching method, we have constructed a system of linearly independent transversally isotropic tensors of rank 5, which contains 51 tensors, as desired. To save space, we do not write them out here. We only note that the linear independence of the tensors thus constructed can be proved easily, and we do not perform this here.

Clearly, a transversally isotropic tensor of rank 5 whose components do not have any symmetry is a linear combination of 51 tensors and hence has 51 linearly independent components. The consideration of various cases of symmetry is not hard, and we omit it.

Now let us construct linearly independent transversally isotropic tensors of rank 6. First, note that it is impossible to construct the desired tensors from the multibases obtained by an odd-order restriction from the three-dimensional multibasis $\mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}_l\mathbf{r}_m\mathbf{r}_n$ of order 6. Indeed, to compose some transversally isotropic tensor from a multibasis obtained by the restriction of the corresponding spatial multibasis, it must contain an even number of basis vectors with capital Latin indices, because only in this case it can be contracted with the two-dimensional Kronecker and Levi-Cività symbols (each of which has two indices). In this case, the multibasis whose number of capital Latin indices is zero, i.e., all of whose indices are equal to 3, is unique in any case. It is called the maximal restriction and is a transversally isotropic tensor (in this case, the even number is zero).

By Theorem 3.4, in the case under study, we must compose all possible multibases by using the even orders (zeroth, second, fourth, and sixth) of restriction of the spatial multibasis $\mathbf{r}_i\mathbf{r}_j\mathbf{r}_k\mathbf{r}_l\mathbf{r}_m\mathbf{r}_n$. First, we compose the multibases that do not contain $\mathbf{r}_3$ (such a basis, called the minimal restriction of the spatial multibasis, is unique: $C_6^0 = 1$) and the multibases containing $\mathbf{r}_3$ two times (the number of such multibases is $C_6^2 = 15$). We have

$$\begin{array}{lllll}
\mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L\mathbf{r}_M\mathbf{r}_N, & \mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L\mathbf{r}_3\mathbf{r}_3, & \mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_3\mathbf{r}_L\mathbf{r}_3, & \mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_K\mathbf{r}_L\mathbf{r}_3, & \mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L\mathbf{r}_3, \\
\mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L\mathbf{r}_3, & \mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3\mathbf{r}_K\mathbf{r}_L, & \mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3\mathbf{r}_K\mathbf{r}_L, & \mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_K\mathbf{r}_L, & \mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_K\mathbf{r}_3\mathbf{r}_L, \\
\mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L, & \mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L, & \mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_3\mathbf{r}_L, & \mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J\mathbf{r}_K\mathbf{r}_L, & \mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_K\mathbf{r}_3\mathbf{r}_L, \\
\mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_3\mathbf{r}_3\mathbf{r}_L. & & & &
\end{array} \tag{3.38}$$

Now one can readily compose the multibases containing $\mathbf{r}_3$ four times. Indeed, such multibases (their number is equal to $C_6^4 = 15$) can be obtained from the multibases containing $\mathbf{r}_3$ two times by replacing the capital Latin indices in them with 3 and by replacing the indices equal to 3 with capital Latin indices. Hence the multibasis that contains $\mathbf{r}_3$ six times and is called the maximal restriction of the spatial multibasis is unique: $C_6^6 = 1$. As a result, we obtain the following multibases:

$$\begin{array}{lllll}
\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_J, & \mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J, & \mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J, & \mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J, & \mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J, \\
\mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3, & \mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3, & \mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3, & \mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3, & \mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3, \\
\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3, & \mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3, & \mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3, & \mathbf{r}_3\mathbf{r}_I\mathbf{r}_3\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3, & \mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_I\mathbf{r}_J\mathbf{r}_3, \\
\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3\mathbf{r}_3. & & & &
\end{array} \tag{3.39}$$

Now from the first multibasis (3.38) we compose all linearly independent two-dimensional transversally isotropic (gyrotropic) tensors of rank 6, whose number, as was shown above, is equal to 20. We note that the set of these tensors contains all linearly independent two-dimensional isotropic tensors of rank 6 obtained from the spatial isotropic tensors (1.5) by the operation of minimal restriction. Hence we have the following statement.

Proposition 3.7. *The two-dimensional isotropic tensors are the minimal restrictions of the corresponding three-dimensional isotropic tensors. The minimal restrictions of linearly independent three-dimensional isotropic tensors are linearly independent two-dimensional isotropic tensors.*

It follows from this proposition that, by using the operation of minimal restriction, we obtain all 15 two-dimensional linearly independent isotropic tensors of rank 6 from the tensors (1.5). By introducing the notation $\underset{\sim}{\mathbb{C}}_{(\alpha)}$ for tensors of rank 6 and by preserving the above notation for the two-dimensional

ETBP and the tensors (3.18) of rank 4, we obtain the following linearly independent two-dimensional isotropic tensors of rank 6:

$$\begin{aligned}
&\underset{\sim}{\mathbb{C}}_{(1)} = \underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}} = \underset{\approx}{\mathbf{C}}_{(1)}\underset{\sim}{\mathbf{I}} = \underset{\sim}{\mathbf{I}}\underset{\approx}{\mathbf{C}}_{(1)}, \quad \underset{\sim}{\mathbb{C}}_{(2)} = \underset{\sim}{\mathbf{I}}\underset{\approx}{\mathbf{C}}_{(2)}, \quad \underset{\sim}{\mathbb{C}}_{(3)} = \underset{\sim}{\mathbf{I}}\underset{\approx}{\mathbf{C}}_{(3)}, \quad \underset{\sim}{\mathbb{C}}_{(4)} = \underset{\approx}{\mathbf{C}}_{(2)}\underset{\sim}{\mathbf{I}},\\
&\underset{\sim}{\mathbb{C}}_{(5)} = \underset{\approx}{\mathbf{C}}_{(3)}\underset{\sim}{\mathbf{I}}, \quad \underset{\sim}{\mathbb{C}}_{(6)} = \mathbf{r}_I\underset{\approx}{\mathbf{C}}_{(1)}\mathbf{r}_I, \quad \underset{\sim}{\mathbb{C}}_{(7)} = \mathbf{r}_I\underset{\approx}{\mathbf{C}}_{(2)}\mathbf{r}_I, \quad \underset{\sim}{\mathbb{C}}_{(8)} = \mathbf{r}_I\underset{\approx}{\mathbf{C}}_{(3)}\mathbf{r}_I,\\
&\underset{\sim}{\mathbb{C}}_{(9)} = \mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_J\mathbf{r}_I\mathbf{r}_J, \quad \underset{\sim}{\mathbb{C}}_{(10)} = \mathbf{r}_I\mathbf{r}_J\underset{\sim}{\mathbf{I}}\mathbf{r}_I\mathbf{r}_J, \quad \underset{\sim}{\mathbb{C}}_{(11)} = \mathbf{r}_I\mathbf{r}_J\mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_J, \quad \underset{\sim}{\mathbb{C}}_{(12)} = \mathbf{r}_I\mathbf{r}_J\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J\mathbf{r}_K,\\
&\underset{\sim}{\mathbb{C}}_{(13)} = \mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_I\mathbf{r}_J\mathbf{r}_K, \quad \underset{\sim}{\mathbb{C}}_{(14)} = \mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J, \quad \underset{\sim}{\mathbb{C}}_{(15)} = \mathbf{r}_I\mathbf{r}_J\mathbf{r}_K\mathbf{r}_J\mathbf{r}_I\mathbf{r}_K.
\end{aligned} \tag{3.40}$$

Since the tensors (3.40) are simultaneously two-dimensional transversally isotropic tensors of rank 6, we have 15 linearly independent two-dimensional transversally isotropic (gyrotropic) tensors of rank 6 in the form (3.40). Five more tensors are required to complete the system of linearly independent tensors. Obviously, for the five missing tensors we can take any five linearly independent two-dimensional transversally isotropic tensors of rank 6 other than (3.40) that, together with the tensors (3.40), form a linearly independent system of 20 tensors. To construct the missing tensors, we consider the tensors

$$\begin{aligned}
&\underset{\sim}{\mathbf{I}} = \mathbf{r}_I\mathbf{r}_I, \quad \underset{\approx}{\mathbf{C}}_{(1)} = \underset{\sim}{\mathbf{I}}\underset{\sim}{\mathbf{I}}, \quad \underset{\approx}{\mathbf{C}}_{(2)} = \mathbf{r}_I\mathbf{r}_J\mathbf{r}_I\mathbf{r}_J, \quad \underset{\approx}{\mathbf{C}}_{(3)} = \mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_I,\\
&\underset{\sim}{\boldsymbol{\varepsilon}} = \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_J, \quad \underset{\approx}{\mathbf{C}}_{(4)} = \varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_J, \quad \underset{\approx}{\mathbf{C}}_{(5)} = \underset{\sim}{\mathbf{I}}\underset{\sim}{\boldsymbol{\varepsilon}}, \quad \underset{\approx}{\mathbf{C}}_{(6)} = \varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_I\mathbf{r}_L.
\end{aligned} \tag{3.41}$$

Note that, using (3.41), we can compose all linearly independent two-dimensional transversally isotropic tensors of rank 6. To this end, it suffices to group the tensors $\underset{\sim}{\mathbf{I}}$ and $\underset{\sim}{\boldsymbol{\varepsilon}}$ with each other and with the other tensors and also, using Proposition 3.4, compose tensors of rank 6. This is one more method for constructing the desired tensors. One can readily see that the tensors

$$\underset{\sim}{\mathbb{C}}_{(16)} = \mathbf{r}_I\underset{\approx}{\mathbf{C}}_{(4)}\mathbf{r}_I, \quad \underset{\sim}{\mathbb{C}}_{(17)} = \mathbf{r}_I\underset{\approx}{\mathbf{C}}_{(5)}\mathbf{r}_I, \quad \underset{\sim}{\mathbb{C}}_{(18)} = \mathbf{r}_I\underset{\approx}{\mathbf{C}}_{(6)}\mathbf{r}_I, \quad \underset{\sim}{\mathbb{C}}_{(19)} = \underset{\sim}{\mathbf{I}}\underset{\approx}{\mathbf{C}}_{(5)}, \quad \underset{\sim}{\mathbb{C}}_{(20)} = \underset{\approx}{\mathbf{C}}_{(4)}\underset{\sim}{\mathbf{I}} \tag{3.42}$$

formed by the above method by using the first and the last three tensors in (3.41) (which, of course, can also be obtained by the contraction and searching method) are linearly independent. Moreover, they, together with the tensors (3.40), form a linearly independent system. Thus, for linearly independent two-dimensional transversally isotropic tensors of rank 6 we can take the tensors (3.41) and (3.42), whose total number is equal to 20.

Now note that from each multibasis (3.38), starting from the second and using the contractions, which would be used to compose the tensors (3.18) by the contraction and searching method (each of them contains four basis vectors with capital Latin indices), we obtain 6 linearly independent transversally isotropic tensors of rank 6 from each multibasis (3.38). Since the number of such bases is equal to 15, we totally obtain $6 \times 15 = 90$ tensors. They are linearly independent. Now we write out some of them. For example, to obtain the desired tensors from the second multibasis in (3.38), it suffices to add $\underset{\sim}{\boldsymbol{\gamma}}^{(3)} = \mathbf{r}_3\mathbf{r}_3$ to the tensors (3.18) on the right. Continuing the numbering of tensors of rank 6, we obtain

$$\begin{aligned}
&\underset{\sim}{\mathbb{C}}_{(21)} = \underset{\approx}{\mathbf{C}}_{(1)}\underset{\sim}{\boldsymbol{\gamma}}^{(3)}, \quad \underset{\sim}{\mathbb{C}}_{(22)} = \underset{\approx}{\mathbf{C}}_{(2)}\underset{\sim}{\boldsymbol{\gamma}}^{(3)}, \quad \underset{\sim}{\mathbb{C}}_{(23)} = \underset{\approx}{\mathbf{C}}_{(3)}\underset{\sim}{\boldsymbol{\gamma}}^{(3)},\\
&\underset{\sim}{\mathbb{C}}_{(24)} = \underset{\approx}{\mathbf{C}}_{(4)}\underset{\sim}{\boldsymbol{\gamma}}^{(3)}, \quad \underset{\sim}{\mathbb{C}}_{(25)} = \underset{\approx}{\mathbf{C}}_{(5)}\underset{\sim}{\boldsymbol{\gamma}}^{(3)}, \quad \underset{\sim}{\mathbb{C}}_{(26)} = \underset{\approx}{\mathbf{C}}_{(6)}\underset{\sim}{\boldsymbol{\gamma}}^{(3)}.
\end{aligned} \tag{3.43}$$

One can readily see that, from the eleventh multibasis in (3.38), we can obtain the tensors that follow from (3.43) if we interchange $\mathbb{C}_{(\alpha)}$, $\alpha = \overline{1,6}$, and $\underset{\sim}{\boldsymbol{\gamma}}^{(3)}$. Here we also write out the tensors that can be obtained from the third multibasis in (3.38):

$$\begin{aligned}
&\underset{\sim}{\mathbb{C}}_{(27)} = \underset{\sim}{\mathbf{I}}\mathbf{r}_K\mathbf{r}_3\mathbf{r}_K\mathbf{r}_3, \quad \underset{\sim}{\mathbb{C}}_{(28)} = \mathbf{r}_I\mathbf{r}_J\mathbf{r}_I\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3, \quad \underset{\sim}{\mathbb{C}}_{(29)} = \mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3,\\
&\underset{\sim}{\mathbb{C}}_{(30)} = \underset{\sim}{\mathbf{I}}\varepsilon_{KL}\mathbf{r}_K\mathbf{r}_3\mathbf{r}_L\mathbf{r}_3, \quad \underset{\sim}{\mathbb{C}}_{(31)} = \varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_I\mathbf{r}_3\mathbf{r}_L\mathbf{r}_3, \quad \underset{\sim}{\mathbb{C}}_{(26)} = \varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\mathbf{I}}\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3.
\end{aligned} \tag{3.44}$$

Note that, by analogy with the last 6 tensors in (3.16), we can compose the following tensors from this multibasis:

$$\begin{aligned}
&\underset{\sim}{\mathbb{A}}_{(1)} = \underset{\sim}{\boldsymbol{\varepsilon}}\varepsilon_{KL}\mathbf{r}_K\mathbf{r}_3\mathbf{r}_L\mathbf{r}_3, \quad \underset{\sim}{\mathbb{A}}_{(2)} = \varepsilon_{IJ}\varepsilon_{KL}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J\mathbf{r}_3\mathbf{r}_L\mathbf{r}_3, \quad \underset{\sim}{\mathbb{A}}_{(3)} = \varepsilon_{IJ}\mathbf{r}_I\underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_3\mathbf{r}_J\mathbf{r}_3,\\
&\underset{\sim}{\mathbb{A}}_{(4)} = \underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_K\mathbf{r}_3\mathbf{r}_K\mathbf{r}_3, \quad \underset{\sim}{\mathbb{A}}_{(5)} = \varepsilon_{IJ}\mathbf{r}_I\mathbf{r}_K\mathbf{r}_J\mathbf{r}_3\mathbf{r}_K\mathbf{r}_3, \quad \underset{\sim}{\mathbb{A}}_{(6)} = \mathbf{r}_I\underset{\sim}{\boldsymbol{\varepsilon}}\mathbf{r}_3\mathbf{r}_I\mathbf{r}_3,
\end{aligned}$$

for which the relations similar to (3.17) have the form

$$\underset{\sim}{\mathbb{A}}_{(1)} = \underset{\sim}{\mathbb{C}}_{(28)} - \underset{\sim}{\mathbb{C}}_{(29)}, \quad \underset{\sim}{\mathbb{A}}_{(2)} = \underset{\sim}{\mathbb{C}}_{(27)} - \underset{\sim}{\mathbb{C}}_{(29)}, \quad \underset{\sim}{\mathbb{A}}_{(3)} = \underset{\sim}{\mathbb{C}}_{(28)} - \underset{\sim}{\mathbb{C}}_{(28)},$$
$$\underset{\sim}{\mathbb{A}}_{(4)} = \underset{\sim}{\mathbb{C}}_{(32)} - \underset{\sim}{\mathbb{C}}_{(31)}, \quad \underset{\sim}{\mathbb{A}}_{(5)} = \underset{\sim}{\mathbb{C}}_{(32)} - \underset{\sim}{\mathbb{C}}_{(30)}, \quad \underset{\sim}{\mathbb{A}}_{(6)} = \underset{\sim}{\mathbb{C}}_{(31)} - \underset{\sim}{\mathbb{C}}_{(30)};$$

i.e., they can be expressed in terms of the tensors (3.44). Quite in a similar way, we can compose the desired tensors from the other multibases in (3.38). Therefore, to save space, we do not construct them here.

Next, for each multibasis in (3.39) except for the last, by contracting them with the two-dimensional Kronecker and Levi-Cività symbols, we can construct two linearly independent transversally isotropic tensors of rank 6 for each of them. Obviously, from 15 multibases we obtain $2 \times 15 = 30$ tensors. The last multibasis (the maximal restriction) is a transversally isotropic tensor of rank 6. Simple calculations show that $20 + 90 + 30 + 1 = 141$ transversally isotropic tensors of rank 6 are composed. The system of these tensors is linearly independent.

Thus, the transversally isotropic tensor of rank 6 whose components do not have any symmetry is a linear combination of a system of linearly independent transversally isotropic tensors of rank 6, which consists of 141 tensors, i.e., such a tensor has 141 linearly independent components. Hence if the tensor components have some symmetry, then the number of independent components of such a tensor is less than 141. It follows from the preceding that, by using the above methods, it is not difficult, if necessary, to construct linearly independent isotropic, orthotropic, and transversally isotropic tensors of any arbitrary rank $n > 6$.

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