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Application of Chebyshev Polynomials to the Theory of Thin Bodies

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Abstract—The application of shifted Chebyshev polynomials of the second kind to the construction of the thin-body theory is considered. Some basic and additional recurrence relations for Chebyshev polynomials are given. Arbitrary-order moments are obtained for the first and second derivatives of a tensor field and for some expressions. Several equations of motion expressed in terms of the moments of displacement and rotation vectors are derived for the moment theory; a number of constitutive relations of the zeroth approximation are obtained.

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The application of shifted orthonormal Chebyshev polynomials to the construction of various variants of the thin-body theory is discussed.

It is well known that the classical orthogonal polynomials (the Legendre polynomials, the Chebyshev polynomials of the first and second kind, etc.) remain orthogonal under linear transformations of orthogonality intervals, i.e., any interval (in particular, the interval $[0,1]$) can be chosen, if needed, as an orthogonality interval. We are interested in the interval $[0,1]$, since like in [2–5] we use a nonclassical (new) parametrization of a thin body region such that the transverse coordinate x^3 belongs to $[0,1]$ and since the tensor fields are expanded in this interval with the aid of Fourier–Chebyshev series in x^3. For this interval the basic recurrence relations for the Chebyshev polynomials of the second kind are given in the literature; these relations are used to obtain some additional relations important for the construction of various variants of the thin-body theory. As a matter of fact, this paper generalizes the Vekua method to the construction of classical and nonclassical thin-body theories with and without consideration of a metric in thickness with the aid of Chebyshev polynomials.

1. BASIC RECURRENCE RELATIONS

The polynomials

$$U_n(x) = [1/(n+1)]\, T'_{n+1}(x), \qquad -1 \le x \le 1, \qquad n \in \mathbb{N}_0 \tag{1}$$

are called the Chebyshev polynomials of the second kind [1, 6] on the interval $[-1,1]$. Here $\mathbb{N}_0$ is the set of natural numbers, $T_n(x) = \cos(n \arccos x)$ are the Chebyshev polynomials of the first kind, $-1 \le x \le 1$, and $n \in \mathbb{N}_0$. In our further discussion, we are interested in shifted Chebyshev polynomials. Based on (1), these polynomials can be defined by the formula

$$U_n^*(t) = U_n(2t-1) = \left\{1/\left[2\sqrt{t(1-t)}\,\right]\right\} \sin[(n+1)\arccos(2t-1)], \qquad 0 \le t \le 1, \qquad n \in \mathbb{N}_0.$$

The functions

$$F^*(r,t) = 1/[(1+r)^2 - 4rt], \qquad h^*(x) = 2\sqrt{t(1-t)}, \qquad |r| \le 1, \qquad 0 \le t \le 1 \tag{2}$$

are the generating and weighting functions for these polynomials, respectively.

The basic relations of interest can be obtained with the aid of the generating function (2) as is done, e.g., in [7] when finding the recurrence relations for the Legendre polynomials. Obviously, these basic relations can first be derived for the polynomials $U_n(x)$ and then for the polynomials $U_n^*(t)$ by substituting $2t-1$, $0 \le t \le 1$, for x. As a result, we come to the following basic relations:

$$\begin{aligned} &4tU_n^*(t) = U_{n-1}^*(t) + 2U_n^*(t) + U_{n+1}^*(t), \qquad 2tU_n^{*\prime}(t) = 2nU_n^*(t) + U_{n-1}^{*\prime}(t) + U_n^{*\prime}(t), \qquad n \ge 1;\\ &U_n^{*\prime}(t) = 4nU_{n-1}^*(t) + U_{n-2}^{*\prime}(t), \qquad n \ge 2. \end{aligned} \tag{3}$$

2. ADDITIONAL RECURRENCE RELATIONS

These additional relations can be obtained on the basis of (3) in the same way as similar relations for the Legendre polynomials [2]. Therefore, here we do not consider their derivation and write down the most important of them:

$$2^{2s}t^sU_k^*(t)=\sum_{p=0}^{2s}C_{2s}^pU_{k-s+p}^*(t),\qquad k-s\geq 0,\qquad k\in\mathbb{N}_0; \tag{4}$$

$$2^{2(k+1)}t^{k+1}U_k^*(t)=\sum_{p=1}^{2k+2}C_{2k+2}^pU_{p-1}^*(t),\qquad k\in\mathbb{N}_0; \tag{5}$$

$$2^{2(k+s)}t^{k+s}U_k^*(t)=-\sum_{q=2}^{s}C_{2k+2s}^{q-2}U_{s-q}^*(t)+\sum_{p=s}^{2k+2s}C_{2k+2s}^pU_{p-s}^*(t),\qquad k\in\mathbb{N}_0,\qquad s\geq 2; \tag{6}$$

$$2^{2s}t^sU_m^*(t)U_n^*(t)=\sum_{p=0}^{m}\sum_{q=0}^{2s}C_{2s}^qU_{n-m-s+2p+q}^*(t),\qquad n-m-s\geq 0; \tag{7}$$

$$U_n^{*'}(t)=4\sum_{k=0}^{[(n-1)/2]}(n-2k)U_{n-(2k+1)}^*(t)=4\sum_{k=0}^{[(n-1)/2]}(2k+1+a)\,U_{2k+a}^*(t),\qquad n\geq 1; \tag{8}$$

$$\begin{aligned}U_n^{*''}(t)&=2^4\sum_{k=0}^{[(n-2)/2]}(k+1)(n-k)[n-(2k+1)]U_{n-(2k+2)}^*(t)\\&=2^2\sum_{k=0}^{[(n-2)/2]}(2k+2-a)\big[(n+1)^2-(2k+2-a)^2\big]U_{2k+1-a}^*(t),\qquad n\geq 2;\end{aligned} \tag{9}$$

$$\begin{aligned}2^{2s}t^sU_n^{*'}(t)&=4\sum_{k=0}^{(n-s-2)/2}\sum_{p=0}^{2s}(n-2k)C_{2s}^pU_{n-s-2k-1+p}^*(t)+4s\sum_{p=1}^{2s}C_{2s}^p\,U_{p-1}^*(t)\\&+4\sum_{k=(n-s+2)/2}^{[(n-1)/2]}(n-2k)\Big[-\sum_{q=2}^{2k+1-(n-s)}C_{2s}^{q-2}U_{2k+1-(n-s)-q}^*(t)+\sum_{p=s}^{2s}C_{2s}^pU_{n-s-2k-1+p}^*(t)\Big],\\&n-s=2l,\qquad l\geq 0,\qquad n\geq 1,\qquad s\geq 0;\end{aligned} \tag{10}$$

$$\begin{aligned}2^{2s}t^sU_n^{*'}&=4\sum_{k=0}^{(s-a-2)/2}(2k+1+a)\Big[-\sum_{q=2}^{s-2k-a}C_{2s}^{q-2}U_{s-2k-a-q}^*+\sum_{p=s-2k-a}^{2s}C_{2s}^p\,U_{p-s+2k+a}^*\Big]\\&+4\sum_{k=(s-a)/2}^{[(n-1)/2]}\sum_{p=0}^{2s}(2k+1+a)C_{2s}^p\,U_{2k+a-s+p}^*,\qquad n-s=2l+1,\qquad l\geq 0,\qquad n\geq 1,\qquad s\geq 0;\end{aligned} \tag{11}$$

$$\begin{aligned}2^{2s}t^sU_n^{*'}&=4\sum_{k=0}^{[(n-1)/2]}(2k+1+a)\Big[-\sum_{q=2}^{s-2k-a}C_{2s}^{q-2}U_{s-2k-a-q}^*+\sum_{p=s-2k-a}^{2s}C_{2s}^pU_{p-(s-2k-a)}^*\Big],\\&s-n\geq 1,\qquad n\geq 1;\end{aligned} \tag{12}$$

$$\begin{aligned}2^{2s}t^sU_n^{*''}(t)&=2^4\sum_{k=0}^{(n-s-2)/2}\sum_{p=0}^{2s}(k+1)(n-k)[n-(2k+1)]C_{2s}^pU_{n-s-2k-2+p}^*(t)\\&+2^4\sum_{k=(n-s)/2}^{[(n-2)/2]}(k+1)(n-k)[n-(2k+1)]\Big[-\sum_{q=2}^{2k+2-n+s}C_{2s}^{q-2}\,U_{2k+2-n+s-q}^*(t)\\&+\sum_{p=2k+2-n+s}^{2s}C_{2s}^p\,U_{n-s-2k-2+p}^*(t)\Big],\qquad n\geq 2,\qquad n-s=2l,\qquad l\geq 0,\qquad s\geq 0;\end{aligned} \tag{13}$$

$$
\begin{aligned}
2^{2s}t^s U_n^{*''} &= 2^2 \sum_{k=0}^{[(n-2)/2]} (2k+2-a)\big[(n+1)^2-(2k+2-a)^2\big]\Big[-\sum_{q=2}^{s-2k-1+a} C_{2s}^{q-2} U^*_{s-2k-1+a-q} \\
&+ \sum_{p=s-2k-1+a}^{2s} C_{2s}^{p} U^*_{2k+1-a-s+p}\Big], \qquad n \ge 2, \qquad s-n \ge 1;
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
2^{2s}t^s U_n^{*''}(t) &= 2^4 \sum_{k=0}^{(n-s-3)/2} \sum_{p=0}^{2s} (k+1)(n-k)[n-(2k+1)] C_{2s}^{p} U^*_{n-s-2k-2+p}(t) \\
&+ 2^2 s\big[(n+1)^2-s^2\big] \sum_{p=1}^{s} C_{2s}^{p}\, U^*_{p-1}(t) + 2^4 \sum_{k=(n-s+1)/2}^{[(n-2)/2]} (k+1)(n-k)[n-(2k+1)] \\
&\times \Big[-\sum_{q=2}^{2k+2+s-n} C_{2s}^{q-2} U^*_{2k+2+s-n-q} + \sum_{p=2k+2+s-n}^{2s} C_{2s}^{p}\, U^*_{n-s-2k-2+p}\Big], \\
& n \ge 2, \qquad n-s = 2l+1, \qquad l \ge 0, \qquad s \ge 0.
\end{aligned} \tag{15}
$$

Here $a = n-1-2\big[\frac{n-1}{2}\big]$ and $[x]$ is the integer part of x. Note that, with the exception of (7), relations (3)–(15) also hold for the orthonormal Chebyshev polynomials $\{\hat{U}_k^*\}_{k=0}^{\infty}$ of the second kind, where $\hat{U}_k^* = ||U_n^*||^{-1} U_k^*$ and $\|U_k^*\| = \sqrt{\pi}/2$ is the norm of U_k^*, and can be proved by induction. Relation (7) can be represented for orthonormal polynomials in the form

$$
2^{2s}t^s \hat{U}_m^*(t)\hat{U}_n^*(t) = \hat{U}_0^* \sum_{p=0}^{m}\sum_{q=0}^{2s} C_{2s}^{q} \hat{U}^*_{n-m-s+2p+q}(t), \qquad n-m-s \ge 0. \tag{16}
$$

Note that a relation for the Chebyshev polynomials of the first kind is given in [6] as an exercise (this relation is similar to (5)). Some relations obtained from (7) for $s=0$ and $m=n-1$ as well as for $s=0$ and $m=n=k-1$ are also given in [6].

3. TO THE MOMENT THEORY WITH RESPECT TO THE SHIFTED ORTHONORMAL CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

Let us consider a tensor field $\mathbb{F}(x^1, x^2, x^3)$ dependent on the coordinates x^1, x^2, x^3 of the thin-body region in the case of its new parametrization [3, 4]. By a thin body we mean a three-dimensional body such that its thickness is less than its width and length and the new parametrization can be applied to the region occupied by this body. For brevity, instead of $\mathbb{F}(x^1, x^2, x^3)$ we will use $\mathbb{F}(x', x^3)$, where $x' = (x^1, x^2)$ and $x^3 \in [0,1]$. In addition we assume that the tensor fields under consideration are smooth enough. For example, $\mathbb{F}(x', x^3) \in C^m(V \cup \partial V)$, where $m \ge 1$, V is the region occupied by the thin body, and ∂V is the boundary of this region. With respect to the coordinate $x^3 \in [0,1]$ and for each fixed point $x' \in \overset{(-)}{S}$, where $\overset{(-)}{S}$ is the inner base surface, the tensor field $\mathbb{F}(x', x^3)$ can be expanded in a series in the shifted orthonormal polynomials $\{\hat{U}_k^*\}_{k=0}^{\infty}$ of the second kind (see [1]). This expansion is of the form

$$
\mathbb{F}(x', x^3) = \sum_{k=0}^{\infty} \overset{(k)}{\mathbb{F}}(x')\hat{U}_k^*(x^3), \qquad x' \in \overset{(-)}{S}, \qquad x^3 \in [0,1], \tag{17}
$$

where $\overset{(k)}{\mathbb{F}}(x')$ is the kth coefficient in the expansion of $\mathbb{F}(x', x^3)$ in the polynomials $\{\hat{U}_k^*\}_{k=0}^{\infty}$.

Definition 1. The integral

$$
\overset{(k)}{\mathbb{M}}(\mathbb{F}) = \int_0^1 \mathbb{F}(x', x^3)\hat{U}_k^*(x^3)h^*(x^3)\, dx^3, \qquad k \in \mathbb{N}_0 \tag{18}
$$

is called the kth order moment of the tensor field $\mathbb{F}(x', x^3)$ with respect to the polynomials $\{\hat{U}_k^*\}_{k=0}^{\infty}$.

It is not difficult to prove that the following assertion is valid.

Assertion 1. *For any tensor fields $\mathbb{F}(x', x^3)$ and $\mathbb{G}(x', x^3)$ and for any functions $\alpha(x')$ and $\beta(x')$, the following relation holds*:

$$\overset{(k)}{\mathbb{M}}\big[\alpha(x')\mathbb{F} + \beta(x')\mathbb{G}\big] = \alpha(x')\overset{(k)}{\mathbb{M}}(\mathbb{F}) + \beta(x')\overset{(k)}{\mathbb{M}}(\mathbb{G}), \qquad k \in \mathbb{N}_0. \tag{19}$$

Corollary. The moment operator is linear.

Assertion 2. *The kth order moment of the tensor field $\mathbb{F}(x', x^3)$ with respect to the polynomials $\{\hat{U}^*_k\}_{k=0}^{\infty}$ is equal to the kth coefficient in the expansion of $\mathbb{F}(x', x^3)$ in these polynomials with respect to x^3*:

$$\overset{(k)}{\mathbb{M}}(\mathbb{F}) = \int\limits_0^1 \mathbb{F}(x', x^3)\hat{U}^*_k(x^3)h^*(x^3)\,dx^3 = \overset{(k)}{\mathbb{F}}(x'), \qquad k \in \mathbb{N}_0. \tag{20}$$

Relation (19) follows from (18), whereas relation (20) can be proved with the aid of (17) and (18) with consideration of the fact that the polynomials $\{\hat{U}^*_k\}_{k=0}^{\infty}$ are orthonormal.

It is easy to prove the following relations:

$$\overset{(k)}{\mathbb{M}}(\partial_i\mathbb{F}) = \begin{cases} \partial_I \overset{(k)}{\mathbb{F}}(x'), & i = I; \\ \overset{(k)}{\mathbb{F}}{}'(x'), & i = 3, \end{cases} \qquad \overset{(k)}{\mathbb{M}}(\partial_i\partial_j\mathbb{F}) = \begin{cases} \partial_I\partial_J \overset{(k)}{\mathbb{F}}(x'), & i = I, \quad j = J; \\ \partial_I \overset{(k)}{\mathbb{F}}{}'(x'), & i = I, \quad j = 3; \\ \overset{(k)}{\mathbb{F}}{}''(x'), & i = j = 3. \end{cases} \tag{21}$$

Here

$$\overset{(k)}{\mathbb{F}}{}' = 2(k+1)\sum_{p=k}^{\infty}\big[1 - (-1)^{k+p}\big]\overset{(p)}{\mathbb{F}}, \qquad \overset{(k)}{\mathbb{F}}{}'' = 2(k+1)\sum_{p=k}^{\infty}(p-k)(k+p+2)\big[1 + (-1)^{k+p}\big]\overset{(p)}{\mathbb{F}}. \tag{22}$$

Relations (21) can be generalized as follows:

$$\overset{(k)}{\mathbb{M}}[P_N(x^3)\partial_i^p\partial_j^q\mathbb{F}] = \begin{cases} \partial_I^p\partial_J^q\overset{(k)}{\mathbb{M}}\big[P_N(x^3)\mathbb{F}\big], & i = I, \quad j = J; \\ \partial_I\big\{\overset{(k)}{\mathbb{M}}\big[P_N(x^3)\mathbb{F}\big]\big\}^{(q)}, & i = I, \quad j = 3; \\ \big\{\overset{(k)}{\mathbb{M}}\big[P_N(x^3)\mathbb{F}\big]\big\}^{(p+q)}, & i = j = 3. \end{cases} \tag{23}$$

Here $P_N(x^3)$ is a polynomial of degree N; k, N, p, $q \in \mathbb{N}_0$; and the notation $\big\{\overset{(k)}{\mathbb{M}}\big[P_N(x^3)\mathbb{F}\big]\big\}^{(m)}$, $m \in \mathbb{N}_0$, means that the derivative is taken m times.

In order to prove the relations in the first rows of (21) and (23), we use relation (18). The relations in the second and third rows of (21) are proved with the aid of (8) and (11), respectively; the relations in the second and third rows of (23) are proved by induction.

Based on relations (4), (5), and (6) and on the last relation in (23), it is not difficult to prove that

$$\begin{aligned} &\overset{(n)}{\mathbb{M}}\big[(x^3)^s\partial_3^m\mathbb{F}\big] = \sum_{p=0}^{2s} 2^{-2s}C_{2s}^p \overset{(n-s+p)}{\mathbb{F}}{}^{(m)}(x'), \qquad n - s \geq 0, \qquad s,\, m \in \mathbb{N}_0; \\ &\overset{(n)}{\mathbb{M}}\big[(x^3)^s\partial_3^m\mathbb{F}\big] = \sum_{p=1}^{2n+2} 2^{-2(n+1)}C_{2n+2}^p \overset{(p-1)}{\mathbb{F}}{}^{(m)}(x'), \qquad s = n+1, \qquad n,\, m \in \mathbb{N}_0; \\ &\overset{(n)}{\mathbb{M}}\big[(x^3)^s\partial_3^m\mathbb{F}\big] = -\sum_{q=2}^{s-n} 2^{-2s}C_{2s}^{q-2} \overset{(s-n-q)}{\mathbb{F}}{}^{(m)} + \sum_{p=s-n}^{2s} 2^{-2s}C_{2s}^p \overset{(n-s+p)}{\mathbb{F}}{}^{(m)}, \qquad s \geq n+2, \qquad n,\, m \in \mathbb{N}_0. \end{aligned} \tag{24}$$

The following relations allow us to find the kth order moment for the product of two functions:

$$
\begin{aligned}
&\overset{(k)}{M}\big[2^{2s}(x^3)^s fg\big] = \hat{U}_0^* \sum_{n=0}^{\infty}\sum_{q=0}^{2s}\sum_{p=0}^{k-s+q} C_{2s}^{q} \overset{(n+p)}{f}\overset{(n+k-s-p+q)}{g}, \qquad k-s\ge 0, \qquad s\ge 0;\\
&\overset{(k)}{M}\big[2^{2(k+1)}(x^3)^{k+1} fg\big] = \hat{U}_0^* \sum_{n=0}^{\infty}\sum_{q=1}^{2(k+1)}\sum_{p=0}^{q-1} C_{2k+2}^{q} \overset{(n+p)}{f}\overset{(n-p-1+q)}{g}, \qquad k\ge 0;\\
&\overset{(k)}{M}\big[2^{2(k+s)}(x^3)^{k+s} fg\big]\\
&\quad = \hat{U}_0^* \sum_{n=0}^{\infty}\Big(-\sum_{q=2}^{s}\sum_{p=0}^{s-q} C_{2(k+s)}^{q-2} \overset{(n+p)}{f}\overset{(n+s-p-q)}{g} + \sum_{q=s}^{2(k+s)}\sum_{p=0}^{q-s} C_{2(k+s)}^{q} \overset{(n+p)}{f}\overset{(n-s-p+q)}{g}\Big), \qquad s\ge 2, \qquad k\ge 0.
\end{aligned} \tag{25}
$$

These relations can be proved with the aid of (4)–(6) and (16). From the first relation in (25), obviously, an expression of the kth order moment for the product of two functions can be obtained for $s=0$.

4. THE EQUATIONS OF MOTION IN TERMS OF THE MOMENTS OF DISPLACEMENT AND ROTATION VECTORS FOR THE NONCLASSICAL (MOMENT) THEORY

In the moment theory of elasticity for nonisothermal processes, as is known [8, 9], the equations in terms of displacements and rotations can be represented in the case of a homogeneous isotropic medium with a symmetry center as follows:

$$
\begin{aligned}
&(\lambda+\mu-\alpha)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + (\mu+\alpha)\Delta\mathbf{u} + 2\alpha\,\mathrm{curl}\,\boldsymbol{\varphi} - b\,\mathrm{grad}\,\theta + \rho\mathbf{F} = \rho\,\frac{\partial^2\mathbf{u}}{\partial t^2},\\
&(\gamma+\delta-\beta)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi} + (\delta+\beta)\Delta\boldsymbol{\varphi} + 2\alpha\,\mathrm{curl}\,\mathbf{u} - 4\alpha\,\boldsymbol{\varphi} + \rho\mathbf{m} = \underset{\sim}{\mathbf{J}}\cdot\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}.
\end{aligned}
$$

Here α, λ, μ, β, γ, and δ are material constants; $\mathbf{u}$ and $\boldsymbol{\varphi}$ are the displacement vector and the rotation vector, respectively; $b = a(3\lambda+2\mu)$; a is the thermal-expansion coefficient; ρ is the density of the medium; $\mathbf{F}$ is the mass force; and $\mathbf{m}$ is the mass moment. In the literature mentioned above, it is assumed that $\underset{\sim}{\mathbf{J}} = J\underset{\sim}{\mathbf{E}}$, where J is a dynamic characteristic and $\underset{\sim}{\mathbf{E}}$ is the second-rank unit tensor.

Using the new parametrization of the thin-body region, we can rewrite these equations in the form

$$
\begin{aligned}
&\Big[\underset{\sim}{\mathbf{M}}^{\bar{M}\bar{N}} g_{\bar{M}}^{P} g_{\bar{N}}^{Q} N_P N_Q + \underset{\sim}{\mathbf{M}}^{\bar{M}\bar{3}} g_{\bar{M}}^{P}(N_P\nabla_3 + \nabla_3 N_P) + \underset{\sim}{\mathbf{M}}^{\bar{3}\bar{3}}\nabla_3^2\Big]\cdot\mathbf{u} + 2\alpha\Big[C^{\bar{M}\bar{N}} g_{\bar{M}}^{P} N_P \varphi_{\bar{N}}\mathbf{r}_{\bar{3}}\\
&\qquad + C^{\bar{L}\bar{M}}\big(g_{\bar{M}}^{P} N_P\varphi_{\bar{3}} - \nabla_3\varphi_{\bar{M}}\big)\mathbf{r}_{\bar{L}}\Big] - b\big(\mathbf{r}^{\bar{M}} g_{\bar{M}}^{P} N_P + \mathbf{r}^{\bar{3}}\nabla_3\big)\theta + \rho\mathbf{F} = \rho\,\frac{\partial^2\mathbf{u}}{\partial t^2},\\
&\Big[\underset{\sim}{\mathbf{L}}^{\bar{M}\bar{N}} g_{\bar{M}}^{P} g_{\bar{N}}^{Q} N_P N_Q + \underset{\sim}{\mathbf{L}}^{\bar{M}\bar{3}} g_{\bar{M}}^{P}(N_P\nabla_3 + \nabla_3 N_P) + \underset{\sim}{\mathbf{L}}^{\bar{3}\bar{3}}\nabla_3^2\Big]\cdot\boldsymbol{\varphi}\\
&\qquad + 2\alpha\Big[C^{\bar{L}\bar{M}}\big(g_{\bar{M}}^{P} N_P u_{\bar{3}} - \nabla_3 u_{\bar{M}}\big)\mathbf{r}_{\bar{L}} + C^{\bar{M}\bar{N}} g_{\bar{M}}^{P} N_P u_{\bar{N}}\mathbf{r}_{\bar{3}}\Big] - 4\alpha\boldsymbol{\varphi} + \rho\,\mathbf{m} = \underset{\sim}{\mathbf{J}}\cdot\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2},
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
&\underset{\sim}{\mathbf{M}}^{\bar{m}\bar{n}} = \underset{\approx}{\mathbf{M}}\cdot\cdot\,\mathbf{r}^{\bar{m}}\mathbf{r}^{\bar{n}}, \qquad \underset{\sim}{\mathbf{L}}^{\bar{m}\bar{n}} = \underset{\approx}{\mathbf{L}}\cdot\cdot\,\mathbf{r}^{\bar{m}}\mathbf{r}^{\bar{n}}, \qquad \underset{\approx}{\mathbf{M}} = \frac{1}{2}(\lambda+\mu-\alpha)\big(\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)}\big) + (\mu+\alpha)\underset{\approx}{\mathbf{C}}_{(1)},\\
&\underset{\approx}{\mathbf{L}} = \frac{1}{2}(\gamma+\delta-\beta)\big(\underset{\approx}{\mathbf{C}}_{(2)} + \underset{\approx}{\mathbf{C}}_{(3)}\big) + (\delta+\beta)\underset{\approx}{\mathbf{C}}_{(1)}, \qquad g_{\bar{M}}^{P} = \sum_{s=0}^{\infty} \underset{(s)}{A}{}_{\overset{+}{M}}^{\bar{P}}(x^3)^s, \qquad g_{\bar{M}}^{P} g_{\bar{N}}^{Q} = \sum_{s=0}^{\infty} \underset{(s)}{B}{}_{\overset{+}{M}\overset{+}{N}}^{\bar{P}\bar{Q}}(x^3)^s,\\
&\underset{(s)}{B}{}_{\overset{+}{M}\overset{+}{N}}^{\bar{P}\bar{Q}} = \sum_{r=0}^{s} \underset{(s-r)}{A}{}_{\overset{+}{M}}^{\bar{P}} \underset{(r)}{A}{}_{\overset{+}{N}}^{\bar{Q}}, \qquad \underset{(s)}{A}{}_{\overset{+}{M}}^{\bar{P}} = \big(g_{\bar{N}_1}^{\bar{P}} - g_{\overset{+}{N}_1}^{\bar{P}}\big)\big(g_{\bar{N}_2}^{\bar{N}_1} - g_{\overset{+}{N}_2}^{\bar{N}_1}\big)\dots\big(g_{\bar{N}_{s-1}}^{\bar{N}_{s-2}} - g_{\overset{+}{N}_{s-1}}^{\bar{N}_{s-2}}\big)\big(g_{\bar{M}}^{\bar{N}_{s-1}} - g_{\overset{+}{M}}^{\bar{N}_{s-1}}\big),\\
&\underset{(0)}{A}{}_{\overset{+}{M}}^{\bar{P}} = g_{\bar{M}}^{\bar{P}}, \qquad N_P = \nabla_P - g_P^{\bar{3}}\nabla_3 = \nabla_P - x^3 g_{\overset{+}{P}}^{\bar{3}}\nabla_3, \qquad g_{\overset{+}{P}}^{3} = \partial_P \ln h, \qquad C^{\bar{M}\bar{N}} = \big(\mathbf{r}^{\bar{M}}\times\mathbf{r}^{\bar{N}}\big)\cdot\mathbf{r}^{\bar{3}},
\end{aligned} \tag{27}
$$

and $\underset{\approx}{\mathbf{C}}_{(1)}$, $\underset{\approx}{\mathbf{C}}_{(2)}$, and $\underset{\approx}{\mathbf{C}}_{(3)}$ are the fourth-rank isotropic tensors [10].

From Eqs. (26) and relations (19), (21), (23), and (27) it follows that, in order to represent (26) in terms of moments, it is necessary to know the moments of the following expressions:

$$\overset{(k)}{\mathbf{M}}\big(g^{P}_{\underset{M}{-}} g^{Q}_{\underset{N}{-}} N_P N_Q \mathbf{u}\big) = \sum_{s=0}^{\infty} \underset{(s)}{B}{}^{\bar{P}\bar{Q}}_{\underset{M}{+}\underset{N}{+}} \Big\{\nabla_P \nabla_Q \overset{(k)}{\mathbf{M}}\big[(x^3)^s \mathbf{u}\big] - \big(g^{\bar{3}}_{\underset{P}{+}} \nabla_Q + g^{\bar{3}}_{\underset{Q}{+}} \nabla_P\big) \overset{(k)}{\mathbf{M}}{}' \big[(x^3)^{s+1} \mathbf{u}\big]$$

$$+ g^{\bar{3}}_{\underset{P}{+}} g^{\bar{3}}_{\underset{Q}{+}} \overset{(k)}{\mathbf{M}}{}'' \big[(x^3)^{s+2} \mathbf{u}\big]\Big\}, \quad \overset{(k)}{\mathbf{M}}\big(g^{P}_{\underset{M}{-}} N_P \nabla_3 \mathbf{u}\big) = \sum_{s=0}^{\infty} \underset{(s)}{A}{}^{\bar{P}}_{\underset{M}{+}} \Big\{\nabla_P \overset{(k)}{\mathbf{M}}{}' \big[(x^3)^s \mathbf{u}\big] - g^{\bar{3}}_{\underset{P}{+}} \overset{(k)}{\mathbf{M}}{}'' \big[(x^3)^{s+1} \mathbf{u}\big]\Big\}, \tag{28}$$

$$\overset{(k)}{M}\big(g^{P}_{\underset{M}{-}} N_P u_{\underset{n}{-}}\big) = \sum_{s=0}^{\infty} \underset{(s)}{A}{}^{\bar{P}}_{\underset{M}{+}} \Big\{\nabla_P \overset{(k)}{M}\big[(x^3)^s u_{\underset{n}{-}}\big] - g^{\bar{3}}_{\underset{P}{+}} \overset{(k)}{u}{}'_{\underset{n}{-}}\Big\}, \quad \overset{(k)}{M}(\nabla_3 u_{\underset{n}{-}}) = \overset{(k)}{u}{}'_{\underset{n}{-}}, \quad \overset{(k)}{\mathbf{M}}(\nabla_3^2 \mathbf{u}) = \overset{(k)}{\mathbf{u}}{}''.$$

Replacing u by φ in (28), we come to similar relations for $\boldsymbol{\varphi}$ and its components $\varphi_{\underset{n}{-}}$. It is clear that we can find the final expressions for (28) on the basis of (21) and (24) for $m = 1$. Taking into account the resulting relations for $\mathbf{u}$ and $u_{\underset{n}{-}}$ and for $\boldsymbol{\varphi}$ and $\varphi_{\underset{n}{-}}$, from (26) we obtain the sought-for equations. Obviously, we come to an infinite system of equations; this system can always be reduced to a finite one with the use of the reduction method discussed in [11]. Contrary to [11], in this paper we use the Chebyshev polynomials of the second kind instead of Legendre polynomials, since for the Chebyshev polynomials it is possible to obtain more compact and general recurrence relations than those for the Legendre polynomials [2]. In addition, the method used in [11] can be generalized to the case of the moment theory for anisotropic materials. Moreover, a metric variation in thickness ($s \neq 0$) can be taken into account; this allows us to choose many methods for the reduction to a finite system, although in practice it is sufficient to restrict ourselves to the systems of equations obtained, e.g., for $s = 0$, 1, 2. This depends on a particular problem and a required accuracy of approximation.

Note that a number of problems are formulated in [12] for the moment theory when the classical parametrization of a thin-body region is used in terms of moments of covariant components of the stress and couple-stress tensors with respect to the Legendre polynomials. In our case, of course, it is not difficult to make similar formulations of these problems.

5. ON CONSTITUTIVE RELATIONS

We restrict ourselves to considering an arbitrary anisotropic linear elastic medium with a symmetry center. In our case the constitutive relations can then be represented in the form

$$\underset{\sim}{\mathbf{P}} = \underset{\simeq}{\mathbf{C}}{}^{\bar{M}} \cdot g^{P}_{\underset{M}{-}} N_P \mathbf{u} + \underset{\simeq}{\mathbf{C}}{}^{\bar{3}} \cdot \partial_3 \mathbf{u} - \underset{\approx}{\mathbf{C}} \cdot\cdot \underset{\simeq}{\mathbf{C}} \cdot \boldsymbol{\varphi} - \underset{\sim}{\mathbf{b}}\theta, \qquad \underset{\sim}{\boldsymbol{\mu}} = \underset{\simeq}{\mathbf{D}}{}^{\bar{M}} \cdot g^{P}_{\underset{M}{-}} N_P \boldsymbol{\varphi} + \underset{\simeq}{\mathbf{D}}{}^{\bar{3}} \cdot \partial_3 \boldsymbol{\varphi}, \tag{29}$$

where $\underset{\simeq}{\mathbf{C}}{}^{\bar{n}} = \underset{\approx}{\mathbf{C}} \cdot \mathbf{r}^{\bar{n}}$, $\underset{\simeq}{\mathbf{D}}{}^{\bar{n}} = \underset{\approx}{\mathbf{D}} \cdot \mathbf{r}^{\bar{n}}$, $\underset{\approx}{\mathbf{C}}$ and $\underset{\approx}{\mathbf{D}}$ are the fourth-rank tensors, $\underset{\simeq}{\mathbf{C}}$ is the third-rank discriminant tensor, and $\underset{\sim}{\mathbf{b}}$ is the second-rank tensor of thermomechanical properties. Note that the first relation in (22) can be rewritten as

$$\overset{(k)}{\mathbb{F}}{}' = 2(k+1)\Big\{\sum_{p=k}^{N} \big[1 - (-1)^{k+p}\big] \overset{(p)}{\mathbb{F}} + \big[\overset{(+)}{\mathbb{F}}{}' - (-1)^k \overset{(-)}{\mathbb{F}}{}'\big]\Big\}, \qquad \overset{(+)}{\mathbb{F}}{}' = \sum_{p=N+1}^{\infty} \overset{(p)}{\mathbb{F}}, \qquad \overset{(-)}{\mathbb{F}}{}' = \sum_{p=N+1}^{\infty} \overset{(p)}{\mathbb{F}}. \tag{30}$$

Definition 2. The constitutive relations of order s are said to be approximate if they are obtained from (29) by retaining the first $s + 1$ terms in the expansion of $g^{P}_{\underset{M}{-}}$.

Let us consider the constitutive relations of the zeroth approximation in terms of moments for a material homogeneous in x^3. By Definition 2, these relations can be found from (29) if $g^{P}_{\underset{M}{-}}$ is replaced by δ^{P}_{M}. Applying the kth order moment operator to the resulting relations and taking into account (19) and the first relations in (21) and (30), we come to the sought-for constitutive relations for $N \geq k \geq 0$:

$$\overset{(k)}{\underset{\sim}{\mathbf{P}}}{}_{(0)} = \overset{(k)}{\underset{\sim}{\mathbf{P}}}{}_{(0,N)} + \underset{\simeq}{\mathbf{A}}{}^{\bar{3}}_{(0,k)} \cdot \overset{(+)}{\mathbf{u}}{}' + \underset{\simeq}{\mathbf{B}}{}^{\bar{3}}_{(0,k)} \cdot \overset{(-)}{\mathbf{u}}{}', \qquad \overset{(k)}{\underset{\sim}{\boldsymbol{\mu}}}{}_{(0)} = \overset{(k)}{\underset{\sim}{\boldsymbol{\mu}}}{}_{(0,N)} + \underset{\simeq}{\mathbf{K}}{}^{\bar{3}}_{(0,k)} \cdot \overset{(+)}{\boldsymbol{\varphi}}{}' + \underset{\simeq}{\mathbf{L}}{}^{\bar{3}}_{(0,k)} \cdot \overset{(-)}{\boldsymbol{\varphi}}{}', \tag{31}$$

where

$$\underset{\sim}{\overset{(k)}{\boldsymbol{\mu}}}_{(0,N)} = \underset{\simeq}{\mathbf{D}}^{\bar{M}} \cdot \left\{ \nabla_M \overset{(k)}{\boldsymbol{\varphi}} - g^{\bar{3}}_{\substack{+\\ M}} [k \overset{(k)}{\boldsymbol{\varphi}} + 2(k+1) \sum_{p=k+1}^{N} \overset{(p)}{\boldsymbol{\varphi}}] \right\} + 2(k+1) \underset{\simeq}{\mathbf{D}}^{\bar{3}} \cdot \left\{ \sum_{p=k}^{N} [1-(-1)^{k+p}] \overset{(p)}{\boldsymbol{\varphi}} \right\},$$

$$\underset{\sim}{\overset{(k)}{\mathbf{P}}}_{(0,N)}(x') = \underset{\simeq}{\mathbf{C}}^{\bar{M}} \cdot \left\{ \nabla_M \overset{(k)}{\mathbf{u}}(x') - g^{\bar{3}}_{\substack{+\\ M}} [k \overset{(k)}{\mathbf{u}}(x') + 2(k+1) \sum_{p=k+1}^{N} \overset{(p)}{\mathbf{u}}(x')] \right\}$$

$$+ 2(k+1) \underset{\simeq}{\mathbf{C}}^{\bar{3}} \cdot \left\{ \sum_{p=k}^{N} [1-(-1)^{k+p}] \overset{(p)}{\mathbf{u}}(x') \right\} - \underset{\approx}{\mathbf{C}} \cdot \underset{\simeq}{\mathbf{C}} \cdot \overset{(k)}{\boldsymbol{\varphi}} - \underset{\sim}{\mathbf{b}} \overset{(k)}{\vartheta}, \qquad N \ge k \ge 0;$$

$$\underset{\simeq}{\mathbf{A}}^{\bar{3}}_{(0,k)} = \underset{\approx}{\mathbf{C}} \cdot \left[2(k+1) \big(\mathbf{r}^{\bar{3}} - g^{\bar{3}}_{\substack{+\\ M}} \mathbf{r}^{\bar{M}} \big) \right], \qquad \underset{\simeq}{\mathbf{B}}^{\bar{3}}_{(0,k)} = \underset{\approx}{\mathbf{C}} \cdot \left[2(k+1)(-1)^{k+1} \mathbf{r}^{\bar{3}} \right]; \quad \mathbf{A} \to \mathbf{K}, \quad \mathbf{C} \to \mathbf{D}, \quad \mathbf{B} \to \mathbf{L}.$$

Here the notation $\mathbf{A} \to \mathbf{K}$, $\mathbf{C} \to \mathbf{D}$, and $\mathbf{B} \to \mathbf{L}$ means that the missing relations can be obtained if $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{B}$ are replaced by $\mathbf{K}$, $\mathbf{D}$, and $\mathbf{L}$, respectively. Note that it is not difficult to derive the constitutive relations in terms of the moments of the sth order approximation. These relations are similar to (31). The above discussion was published in [13] with minor changes. In this paper we corrected some mistakes detected in [13].

The expressions for $\overset{(+)}{\mathbf{u}}{}'$, $\overset{(-)}{\mathbf{u}}{}'$, $\overset{(+)}{\boldsymbol{\varphi}}{}'$, and $\overset{(-)}{\boldsymbol{\varphi}}{}'$ can be derived with the aid of static boundary conditions given on the face surfaces of a thin body [14]. These expressions can also be obtained from the second and third relations of (30) if we replace $\mathbb{F}$ by $\mathbf{u}$ and by $\boldsymbol{\varphi}$, respectively. The bounded and unbounded dynamic problems for the moment thermomechanics of deformable thin solid bodies and the nonstationary temperature problem are formulated in [14] in terms of moments of (r, N) approximation. The above constitutive relations are valid only for a medium homogeneous in x^3; using (25), however, it is not difficult to obtain similar relations for an inhomogeneous medium and, hence, to derive the corresponding equations of motion in terms of moments on the basis of the three-dimensional equations of motion in terms of the stress and couple-stress tensors. It should be noted that the above and other recurrence relations can be used to come to the corresponding equations and the corresponding formulations of problems for the thin-body theory of arbitrary approximation on the basis of any equation of deformable solid body mechanics and any formulation of problems (in particular, on the basis of Pobedria's formulation [15, 16]).

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