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Shell Theory Equations Consistent with Boundary Conditions at Face Surfaces

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Abstract—Shell theory equations consistent with physically meaningful boundary conditions at face surfaces are obtained from the three-dimensional deformable-body mechanics equations related to the reference and actual configurations.

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1. PARAMETRIZATION OF A SHELL DOMAIN

The classical parametrization of a shell domain [1–3] is considered. Here we use the notation introduced in [4] and the standard rules of tensor analysis [2, 3, 5]. As a base surface, we consider a regular surface S relative to which the shell domain is situated asymmetrically [1]. In this case the radius vector of an arbitrary point of the shell can be specified by the relations

$$\hat{\mathbf{r}}(x^1, x^2, x^3) = \mathbf{r}(x^1, x^2) + x^3\mathbf{n}(x^1, x^2), \qquad -\overset{(-)}{h}(x^1, x^2) \le x^3 \le \overset{(+)}{h}(x^1, x^2), \tag{1}$$

where $\mathbf{r}(x^1, x^2)$ is the radius vector of a point on the base surface S, $\mathbf{n}(x^1, x^2)$ is the unit normal vector to the base surface S, x^1 and x^2 are the Gaussian coordinates on S, and x^3 is the transverse coordinate; note that S is not a middle surface. Obviously, $h(x^1, x^2) = \overset{(-)}{h}(x^1, x^2) + \overset{(+)}{h}(x^1, x^2)$ is the thickness of the shell at the point $(x^1, x^2) \in S$. In what follows we assume that $\overset{(-)}{h}(x^1, x^2)$ and $\overset{(+)}{h}(x^1, x^2)$ are piecewise smooth functions of x^1 and x^2.

For $x^3 = -\overset{(-)}{h}(x^1, x^2)$, the first relation in (1) defines the surface $\overset{(-)}{S}$ called the inner surface of the shell; for $x^3 = \overset{(+)}{h}(x^1, x^2)$, this relation defines the surface $\overset{(+)}{S}$ called the outer surface of the shell. The surfaces $\overset{(-)}{S}$ and $\overset{(+)}{S}$ are also called the face surfaces of the shell. Let

$$\overset{(-)}{\mathbf{r}}(x^1, x^2) = \hat{\mathbf{r}}(x^1, x^3, -\overset{(-)}{h}), \qquad \overset{(+)}{\mathbf{r}}(x^1, x^2) = \hat{\mathbf{r}}(x^1, x^3, \overset{(+)}{h});$$

then the vector parametric equations of the surfaces $\overset{(-)}{S}$ and $\overset{(+)}{S}$ can be respectively represented in the form

$$\overset{(-)}{\mathbf{r}}(x^1, x^2) = \mathbf{r}(x^1, x^2) - \overset{(-)}{h}(x^1, x^2)\mathbf{n}(x^1, x^2), \qquad \overset{(+)}{\mathbf{r}}(x^1, x^2) = \mathbf{r}(x^1, x^2) + \overset{(+)}{h}(x^1, x^2)\mathbf{n}(x^1, x^2). \tag{2}$$

Below we consider some physically meaningful boundary conditions at these face surfaces. To accomplish this, it is necessary to determine the unit normal vectors at the points of the face surfaces. Let us find the corresponding expressions for the covariant basis vectors on these surfaces. Let $\mathbf{r}_{\underset{I}{-}} = \partial_I \overset{(-)}{\mathbf{r}} = \dfrac{\partial \overset{(-)}{\mathbf{r}}}{\partial x^I}$ and $\mathbf{r}_{\underset{I}{+}} = \partial_I \overset{(+)}{\mathbf{r}} = \dfrac{\partial \overset{(+)}{\mathbf{r}}}{\partial x^I}$; then from (2) we come to the following relations for the covariant basis vectors of $\overset{(-)}{S}$ and $\overset{(+)}{S}$:

$$\mathbf{r}_{\underset{I}{-}} = (g_I^J + \overset{(-)}{h} b_I^J)\mathbf{r}_J - \partial_I \overset{(-)}{h}\mathbf{n}, \qquad \mathbf{r}_{\underset{I}{+}} = (g_I^J - \overset{(+)}{h} b_I^J)\mathbf{r}_J + \partial_I \overset{(+)}{h}\mathbf{n}. \tag{3}$$

Here the Weingarten formulas $\mathbf{n}_I = -b^J_I \mathbf{r}_J$ are taken into account and b^J_I are the components of the second tensor $\underset{\sim}{\mathbf{b}}$ of the surface S. It is easy to show that the translation components $g^J_{\hat I} = \mathbf{r}_{\hat I} \cdot \mathbf{r}^J$ and $g^{\hat I}_J = \mathbf{r}^{\hat I} \cdot \mathbf{r}_J$ of the second-rank unit tensor (here $\mathbf{r}_{\hat I} = \partial_I \hat{\mathbf{r}}$ and $\mathbf{r}^{\hat I} = \sqrt{\hat g^{-1}}\, \epsilon^{IJ} \mathbf{r}_{\hat J} \times \mathbf{n}$) can be represented in the form (see [4])

$$g^J_{\hat I} = g^J_I - x^3 b^J_I, \qquad g^{\hat I}_J = \hat\vartheta^{-1} A^{\hat I}_J = \hat\vartheta^{-1}[(1 - 2Hx^3) g^I_J + x^3 b^I_J], \tag{4}$$

$$A^{\hat I}_J = \epsilon^{IK} \epsilon_{JL} g^L_{\hat K} = (1 - 2Hx^3) g^I_J + x^3 b^I_J, \qquad \hat\vartheta = \sqrt{\hat g g^{-1}} = \frac{1}{2}\, \epsilon^{IJ} \epsilon_{KL} g^K_{\hat I} g^L_{\hat J} = 1 - 2Hx^3 + K(x^3)^2, \tag{5}$$

where $\sqrt{\hat g} = (\mathbf{r}_{\hat 1} \times \mathbf{r}_{\hat 2}) \cdot \mathbf{n}$, $\sqrt{g} = (\mathbf{r}_1 \times \mathbf{r}_2) \cdot \mathbf{n}$, ϵ^{IJ} and ϵ_{KL} are the Levi-Civita symbols, $H = 2^{-1} I_1(\underset{\sim}{\mathbf{b}})$ is the mean curvature, and $K = \det(\underset{\sim}{\mathbf{b}})$ is the Gaussian curvature of the base surface S. By virtue of the first relation in (4), we have

$$g^J_{\overset{-}{I}} = g^J_{\hat I}\Big|_{x^3 = -\overset{(-)}{h}} = g^J_I + \overset{(-)}{h} b^J_I, \qquad g^J_{\overset{+}{I}} = g^J_{\hat I}\Big|_{x^3 = \overset{(+)}{h}} = g^J_I - \overset{(+)}{h} b^J_I. \tag{6}$$

Let $g^3_{\overset{-}{I}} = -\partial_I \overset{(-)}{h}$ and $g^3_{\overset{+}{I}} = \partial_I \overset{(+)}{h}$. Taking into account (6), we rewrite (3) in the following concise form:

$$\mathbf{r}_{\overset{-}{I}} = g^k_{\overset{-}{I}} \mathbf{r}_k, \qquad \mathbf{r}_{\overset{+}{I}} = g^k_{\overset{+}{I}} \mathbf{r}_k, \qquad \mathbf{r}_3 = \mathbf{n}. \tag{7}$$

Now it is not difficult to find the expressions for the unit normal vectors $\overset{(-)}{\mathbf{n}}$ and $\overset{(+)}{\mathbf{n}}$ to the surfaces $\overset{(-)}{S}$ and $\overset{(+)}{S}$:

$$\overset{(-)}{\mathbf{n}} = -(\mathbf{r}_{\overset{-}{1}} \times \mathbf{r}_{\overset{-}{2}})(|\mathbf{r}_{\overset{-}{1}} \times \mathbf{r}_{\overset{-}{2}}|)^{-1}, \qquad \overset{(+)}{\mathbf{n}} = (\mathbf{r}_{\overset{+}{1}} \times \mathbf{r}_{\overset{+}{2}})(|\mathbf{r}_{\overset{+}{1}} \times \mathbf{r}_{\overset{+}{2}}|)^{-1}. \tag{8}$$

Based on (7), indeed, we represent the vector products in the numerators of (8) as follows:

$$\begin{aligned} \mathbf{r}_{\overset{-}{1}} \times \mathbf{r}_{\overset{-}{2}} &= \sqrt{g}\Big(\epsilon^{IJ} \epsilon_{KL} g^3_{\overset{-}{I}} g^K_{\overset{-}{J}} \mathbf{r}^L + \frac{1}{2}\, \epsilon^{IJ} \epsilon_{KL} g^K_{\overset{-}{I}} g^L_{\overset{-}{J}} \mathbf{n}\Big), \\ \mathbf{r}_{\overset{+}{1}} \times \mathbf{r}_{\overset{+}{2}} &= \sqrt{g}\Big(\epsilon^{IJ} \epsilon_{KL} g^3_{\overset{+}{I}} g^K_{\overset{+}{J}} \mathbf{r}^L + \frac{1}{2}\, \epsilon^{IJ} \epsilon_{KL} g^K_{\overset{+}{I}} g^L_{\overset{+}{J}} \mathbf{n}\Big). \end{aligned} \tag{9}$$

From (5) and the second relation of (4) it follows that

$$\begin{aligned} &g^{\overset{-}{I}}_L = g^{\hat I}_L\Big|_{x^3 = -\overset{(-)}{h}} = \overset{(-)}{\vartheta}{}^{-1} A^{\overset{-}{I}}_L, \qquad g^{\overset{+}{I}}_L = g^{\hat I}_L\Big|_{x^3 = \overset{(+)}{h}} = \overset{(+)}{\vartheta}{}^{-1} A^{\overset{+}{I}}_L, \\ &A^{\overset{-}{I}}_L = A^{\hat I}_L\Big|_{x^3 = -\overset{(-)}{h}} = \epsilon^{IJ} \epsilon_{LK} g^K_{\overset{-}{J}}, \qquad A^{\overset{+}{I}}_L = A^{\hat I}_L\Big|_{x^3 = \overset{(+)}{h}} = \epsilon^{IJ} \epsilon_{LK} g^K_{\overset{+}{J}}, \\ &\overset{(-)}{\vartheta}{}^{-1} = \hat\vartheta\Big|_{x^3 = -\overset{(-)}{h}} = \frac{1}{2}\, \epsilon^{IJ} \epsilon_{KL} g^K_{\overset{-}{I}} g^L_{\overset{-}{J}}, \qquad \overset{(+)}{\vartheta}{}^{-1} = \hat\vartheta\Big|_{x^3 = \overset{(+)}{h}} = \frac{1}{2}\, \epsilon^{IJ} \epsilon_{KL} g^K_{\overset{+}{I}} g^L_{\overset{+}{J}}. \end{aligned} \tag{10}$$

Taking into account (10), from (9) we obtain

$$\begin{aligned} \mathbf{r}_{\overset{-}{1}} \times \mathbf{r}_{\overset{-}{2}} &= \sqrt{\overset{(-)}{g}}\left(\mathbf{n} - g^3_{\overset{-}{I}} g^{\overset{-}{I}}_J \mathbf{r}^J\right), \qquad \sqrt{\overset{(-)}{g}} = \sqrt{g}\, \overset{(-)}{\vartheta}, \\ \mathbf{r}_{\overset{+}{1}} \times \mathbf{r}_{\overset{+}{2}} &= \sqrt{\overset{(+)}{g}}\left(\mathbf{n} - g^3_{\overset{+}{I}} g^{\overset{+}{I}}_J \mathbf{r}^J\right), \qquad \sqrt{\overset{(+)}{g}} = \sqrt{g}\, \overset{(+)}{\vartheta}. \end{aligned} \tag{11}$$

From these relations we get

$$|\mathbf{r}_{\overset{-}{1}} \times \mathbf{r}_{\overset{-}{2}}| = \sqrt{\overset{(-)}{g}} \sqrt{1 + g^3_{\overset{-}{I}} g^3_{\overset{-}{J}} g^{\overset{-}{I}}_K g^{\overset{-}{J}}_L g^{KL}}, \qquad |\mathbf{r}_{\overset{+}{1}} \times \mathbf{r}_{\overset{+}{2}}| = \sqrt{\overset{(+)}{g}} \sqrt{1 + g^3_{\overset{+}{I}} g^3_{\overset{+}{J}} g^{\overset{+}{I}}_K g^{\overset{+}{J}}_L g^{KL}}. \tag{12}$$

By virtue of (11) and (12), from (8) we come to the final expressions for $\overset{(-)}{\mathbf{n}}$ and $\overset{(+)}{\mathbf{n}}$:

$$\overset{(-)}{\mathbf{n}} = -\left(\mathbf{n} - g^3_{\overset{-}{I}} g^{\overset{-}{I}}_J \mathbf{r}^J\right)\left(1 + g^3_{\overset{-}{I}} g^3_{\overset{-}{J}} g^{\overset{-}{I}}_K g^{\overset{-}{J}}_L g^{KL}\right)^{-\frac{1}{2}}, \qquad \overset{(+)}{\mathbf{n}} = \left(\mathbf{n} - g^3_{\overset{+}{I}} g^{\overset{+}{I}}_J \mathbf{r}^J\right)\left(1 + g^3_{\overset{+}{I}} g^3_{\overset{+}{J}} g^{\overset{+}{I}}_K g^{\overset{+}{J}}_L g^{KL}\right)^{-\frac{1}{2}}. \tag{13}$$

Note that similar formulas were obtained in [1].

2. SHELL THEORY EQUATIONS

The shell theory equations can be derived by various ways, e.g., on the basis of the general postulates of mechanics or from three-dimensional equations. As in [1], here we use the second approach but under the assumption that the thickness of a shell is not constant and the base surface does not coincide with the middle one.

It is well known [3] that the three-dimensional deformable-body mechanics equations related to the reference and actual configurations can be represented in the form

$$\overset{\circ}{\nabla}\cdot\overset{\circ}{\underset{\sim}{\mathbf{P}}}+\overset{\circ}{\rho}\mathbf{F}=\overset{\circ}{\rho}\,\frac{\partial^2\mathbf{u}}{\partial t^2},\qquad \nabla\cdot\,\underset{\sim}{\mathbf{P}}+\rho\mathbf{F}=\rho\,\frac{\partial^2\mathbf{u}}{\partial t^2},\tag{14}$$

where $\underset{\sim}{\mathbf{P}}=\underset{\sim}{\mathbf{P}}^T$ is the Cauchy stress tensor, $\overset{\circ}{\underset{\sim}{\mathbf{P}}}=\sqrt{g\overset{\circ}{g}{}^{-1}}\nabla\overset{\circ}{\mathbf{r}}{}^T\cdot\underset{\sim}{\mathbf{P}}$ is the Piola stress tensor, $\sqrt{g}=(\mathbf{r}_1\times\mathbf{r}_2)\cdot\mathbf{r}_3$, $\sqrt{\overset{\circ}{g}}=(\overset{\circ}{\mathbf{r}}_1\times\overset{\circ}{\mathbf{r}}_2)\cdot\overset{\circ}{\mathbf{r}}_3$, and $\overset{\circ}{\nabla}$ and ∇ are the Hamilton operators in the reference and actual configurations. These equations have the same form; therefore, first we consider the equations of the actual configuration, i.e., the second relation in (14). It is not difficult to note that this vector equation can be rewritten as

$$\frac{1}{\sqrt{g}}\,\partial_I(\sqrt{g}\hat{\vartheta}\mathbf{P}^{\hat{I}})+\partial_3(\hat{\vartheta}\mathbf{P}^3)+\hat{\rho}\hat{\vartheta}\hat{\mathbf{F}}=\hat{\rho}\hat{\vartheta}\,\frac{\partial^2\mathbf{u}}{\partial t^2}.\tag{15}$$

Integrating (15) between $-\overset{(-)}{h}(x^1,x^2)$ and $\overset{(+)}{h}(x^1,x^2)$ and taking into account the formula

$$\int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\partial_I\mathbf{A}^{\hat{K}}dx^3=\partial_I\int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\mathbf{A}^{\hat{K}}dx^3-\overset{(+)}{\mathbf{A}}{}^{\overset{+}{K}}\partial_I\overset{(+)}{h}+\overset{(-)}{\mathbf{A}}{}^{\overset{-}{K}}\partial_I\overset{(-)}{h},\tag{16}$$

we obtain $\dfrac{1}{\sqrt{g}}\,\partial_I(\sqrt{g}\mathbf{T}^I)+\mathbf{X}(x^1,x^2)=\mathbf{a}(x^1,x^2)$, or

$$\overset{0}{\nabla}_I\mathbf{T}^I+\mathbf{X}(x^1,x^2)=\mathbf{a}(x^1,x^2),\tag{17}$$

where $\overset{0}{\nabla}_I$ is the surface operator of covariant differentiation and

$$\mathbf{T}^I=\int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\hat{\vartheta}\mathbf{P}^{\hat{I}}dx^3,\qquad \mathbf{p}(x^1,x^2)=\int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\hat{\rho}\hat{\vartheta}\hat{\mathbf{F}}\,dx^3,\qquad \mathbf{a}(x^1,x^2)=\int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}}\hat{\rho}\hat{\vartheta}\,\frac{\partial^2\mathbf{u}}{\partial t^2}\,dx^3,\tag{18}$$

$$\mathbf{X}(x^1,x^2)=\mathbf{p}(x^1,x^2)+\overset{(+)}{\vartheta}\left(\overset{(+)}{\mathbf{P}}{}^3-g^3_{\underset{I}{+}}g^{\overset{+}{I}}_J\overset{(+)}{\mathbf{P}}{}^J\right)-\overset{(-)}{\vartheta}\left(\overset{(-)}{\mathbf{P}}{}^3-g^3_{\underset{I}{-}}g^{\overset{-}{I}}_J\overset{(-)}{\mathbf{P}}{}^J\right).\tag{19}$$

After the vector multiplication of (15) from left by $x^3\mathbf{n}$ and some simple transformations, we come to the relation

$$\begin{aligned}&\frac{1}{\sqrt{g}}\,\partial_I\big(\sqrt{g}\,\mathbf{n}\times(\hat{\vartheta}\mathbf{P}^{\hat{I}}x^3)\big)+\mathbf{r}_I\times(\hat{\vartheta}\mathbf{P}^{\hat{I}})\\&\qquad-\hat{\vartheta}\mathbf{r}_{\hat{p}}\times\hat{\mathbf{P}}^{\hat{p}}+\partial_3(x^3\mathbf{n}\times\hat{\vartheta}\mathbf{P}^3)+\mathbf{n}\times(\hat{\rho}\hat{\vartheta}\hat{\mathbf{F}}x^3)=\mathbf{n}\times\left(\hat{\rho}\hat{\vartheta}\,\frac{\partial^2\mathbf{u}}{\partial t^2}x^3\right).\end{aligned}\tag{20}$$

When deriving (20), we took into account the relation $x^3\mathbf{n}_I=\mathbf{r}_{\hat{I}}-\mathbf{r}_I$. Integrating (20) between $-\overset{(-)}{h}$ and $\overset{(+)}{h}$ and using (16), we obtain $\dfrac{1}{\sqrt{g}}\,\partial_I(\sqrt{g}\mathbf{M}^I)+\mathbf{r}_I\times\mathbf{T}^I+\underset{\approx}{\mathbf{C}}\cdot\cdot\underset{\sim}{\mathbf{T}}+\mathbf{Y}(x^1,x^2)=\mathbf{b}(x^1,x^2)$, or

$$\overset{0}{\nabla}_I\mathbf{M}^I+\mathbf{r}_I\times\mathbf{T}^I+\underset{\approx}{\mathbf{C}}\cdot\cdot\underset{\sim}{\mathbf{T}}+\mathbf{Y}(x^1,x^2)=\mathbf{b}(x^1,x^2),\tag{21}$$

where

$$
\begin{gathered}
\mathbf{M}^I = \mathbf{n} \times \int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}} \hat{\vartheta}\mathbf{P}^{\hat{I}} x^3\, dx^3, \qquad \mathbf{q}(x^1, x^2) = \mathbf{n} \times \int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}} \hat{\rho}\hat{\vartheta}\hat{\mathbf{F}} x^3\, dx^3, \\
\mathbf{b}(x^1, x^2) = \mathbf{n} \times \int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}} \hat{\rho}\hat{\vartheta}\, \frac{\partial^2 \mathbf{u}}{\partial t^2}\, x^3\, dx^3, \qquad \underset{\sim}{\mathbf{T}} = \int\limits_{-\overset{(-)}{h}}^{\overset{(+)}{h}} \hat{\vartheta}\mathbf{P}\, dx^3,
\end{gathered}
\tag{22}
$$

$$
\mathbf{Y}(x^1, x^2) = \mathbf{q}(x^1, x^2) + \mathbf{n} \times \left[\overset{(+)}{h} \overset{(+)}{\vartheta} \left(\overset{(+)}{\mathbf{P}}{}^3 - g^3_{\underset{+}{I}} \overset{+}{g}{}^I_J \overset{(+)}{\mathbf{P}}{}^J \right) \right] - \mathbf{n} \times \left[\overset{(-)}{h} \overset{(-)}{\vartheta} \left(\overset{(-)}{\mathbf{P}}{}^3 - g^3_{\underset{-}{I}} \overset{-}{g}{}^I_J \overset{(-)}{\mathbf{P}}{}^J \right) \right]. \tag{23}
$$

Here $\underset{\approx}{\mathbf{C}}$ is the so-called third-rank discriminant tensor [2] and $\underset{\sim}{\mathbf{T}}$ is the extended force tensor. By

$$
\underset{\sim}{\mathbf{T}}_0 = \mathbf{r}_I \mathbf{T}^I = T^{Ik} \mathbf{r}_I \mathbf{r}_k, \qquad \underset{\sim}{\mathbf{M}} = \mathbf{r}_I \mathbf{M}^I = M^{IJ} \mathbf{r}_I \mathbf{r}_J \tag{24}
$$

we denote the force tensor and the force-moment tensor (or the moment tensor), respectively; then we can represent Eqs. (17) and (21) in the form

$$
\nabla^0 \cdot \underset{\sim}{\mathbf{T}}_0 + \mathbf{X}(x^1, x^2) = \mathbf{a}(x^1, x^2), \qquad \nabla^0 \cdot \underset{\sim}{\mathbf{M}} - \underset{\approx}{\mathbf{C}} \cdot\cdot \underset{\sim}{\mathbf{T}}_0 + \underset{\approx}{\mathbf{C}} \cdot\cdot \underset{\sim}{\mathbf{T}} + \mathbf{Y}(x^1, x^2) = \mathbf{b}(x^1, x^2), \tag{25}
$$

where $\nabla^0 = \mathbf{r}^I \partial_I$ is the surface nabla operator.

In the case of the actual configuration, the third term in the left-hand side of (20) becomes zero, since the Cauchy stress tensor is symmetric; hence, the third term in the left-hand side of the second relation in (25) also becomes zero. In the actual configuration, therefore, the equations of motion take the form

$$
\nabla^0 \cdot \underset{\sim}{\mathbf{T}}_0 + \mathbf{X}(x^1, x^2) = \mathbf{a}(x^1, x^2), \qquad \nabla^0 \cdot \underset{\sim}{\mathbf{M}} + \underset{\approx}{\mathbf{C}} \cdot\cdot \underset{\sim}{\mathbf{T}}{}_0^T + \mathbf{Y}(x^1, x^2) = \mathbf{b}(x^1, x^2). \tag{26}
$$

In the reference configuration, obviously, these equations can be represented as follows:

$$
\overset{\circ}{\nabla}{}^0 \cdot \overset{\circ}{\underset{\sim}{\mathbf{T}}}_0 + \overset{\circ}{\mathbf{X}}(x^1, x^2) = \overset{\circ}{\mathbf{a}}(x^1, x^2), \qquad \overset{\circ}{\nabla}{}^0 \cdot \overset{\circ}{\underset{\sim}{\mathbf{M}}} + \overset{\circ}{\underset{\approx}{\mathbf{C}}} \cdot\cdot (\overset{\circ}{\underset{\sim}{\mathbf{T}}} - \overset{\circ}{\underset{\sim}{\mathbf{T}}}_0) + \overset{\circ}{\mathbf{Y}}(x^1, x^2) = \overset{\circ}{\mathbf{b}}(x^1, x^2). \tag{27}
$$

Here $\overset{\circ}{\nabla}{}^0 = \overset{\circ}{\mathbf{r}}{}^I \partial_I$ is the surface nabla operator in the reference configuration and the quantities in (26) are specified with the use of (18), (19), (22), and (23) if in these relations the quantities are denoted by circles on top.

It should be noted that $\mathbf{X}$ and $\mathbf{Y}$ in Eqs. (26) are expressed in terms of the unknowns $\overset{(-)}{\mathbf{P}}{}^m$ and $\overset{(+)}{\mathbf{P}}{}^n$, whereas $\overset{\circ}{\mathbf{X}}$ and $\overset{\circ}{\mathbf{Y}}$ in Eqs. (27) are expressed in terms of the unknowns $\overset{\circ}{\overset{(-)}{\mathbf{P}}}{}^m$ and $\overset{\circ}{\overset{(+)}{\mathbf{P}}}{}^n$. In order to find them, we use the following physically meaningful boundary conditions (see [1]):

$$
\overset{(-)}{\mathbf{n}} \cdot \overset{(-)}{\underset{\sim}{\mathbf{P}}} = \overset{(-)}{\mathbf{P}}(x^1, x^2), \qquad \overset{(+)}{\mathbf{n}} \cdot \overset{(+)}{\underset{\sim}{\mathbf{P}}} = \overset{(+)}{\mathbf{P}}(x^1, x^2). \tag{28}
$$

Here $\overset{(-)}{\mathbf{P}}(x^1, x^2)$ and $\overset{(+)}{\mathbf{P}}(x^1, x^2)$ are the given vectors of external stress forces on the inner surface $\overset{(-)}{S}$ and on the outer surface $\overset{(+)}{S}$, respectively. By virtue of (13) and (28), we obtain

$$
\overset{(-)}{\mathbf{P}}{}^3 - g^3_{\underset{-}{I}} \overset{-}{g}{}^I_J \overset{(-)}{\mathbf{P}}{}^J = -\sqrt{1 + g^3_{\underset{-}{I}} g^3_{\underset{-}{J}} \overset{-}{g}{}^I_K \overset{-}{g}{}^J_L g^{KL}}\, \overset{(-)}{\mathbf{P}}, \qquad \overset{(+)}{\mathbf{P}}{}^3 - g^3_{\underset{+}{I}} \overset{+}{g}{}^I_J \overset{(+)}{\mathbf{P}}{}^J = \sqrt{1 + g^3_{\underset{+}{I}} g^3_{\underset{+}{J}} \overset{+}{g}{}^I_K \overset{+}{g}{}^J_L g^{KL}}\, \overset{(+)}{\mathbf{P}}. \tag{29}
$$

Taking into account (29), from (19) and (23) we get

$$
\begin{gathered}
\mathbf{X}(x^1, x^2) = \mathbf{p}(x^1, x^2) + \overset{(-)}{C}(x^1, x^2) \overset{(-)}{\mathbf{P}}(x^1, x^2) + \overset{(+)}{C}(x^1, x^2) \overset{(+)}{\mathbf{P}}(x^1, x^2), \\
\mathbf{Y}(x^1, x^2) = \mathbf{q}(x^1, x^2) + \overset{(-)}{C}(x^1, x^2) \overset{(-)}{h} \overset{(-)}{\mathbf{n}} \times \overset{(-)}{\mathbf{P}}(x^1, x^2) + \overset{(+)}{C}(x^1, x^2) \overset{(+)}{h} \overset{(+)}{\mathbf{n}} \times \overset{(+)}{\mathbf{P}}(x^1, x^2),
\end{gathered}
\tag{30}
$$

where

$$\overset{(-)}{C}(x^1, x^2) = \overset{(-)}{\vartheta}\sqrt{1 + g^3_{\overset{-}{I}} g^3_{\overset{-}{J}} g^{\overset{-}{I}}_{K} g^{\overset{-}{J}}_{L} g^{KL}}, \qquad \overset{(-)}{C}(x^1, x^2) = \overset{(+)}{\vartheta}\sqrt{1 + g^3_{\overset{+}{I}} g^3_{\overset{+}{J}} g^{\overset{+}{I}}_{K} g^{\overset{+}{J}}_{L} g^{KL}}.$$

Equations (26) in which $\mathbf{X}$ and $\mathbf{Y}$ are specified by (30) in terms of volume loads and the loads on the face surfaces are the sought-for shell theory equations in the actual configuration.

Similarly, Eqs. (27) in which $\overset{\circ}{\mathbf{X}}$ and $\overset{\circ}{\mathbf{Y}}$ are specified by (30) with the quantities denoted by circles on top are the shell theory equations in the reference configuration. These equations are consistent with the boundary conditions at the face surfaces.

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